Structure and Properties of Traces for Functional Programs

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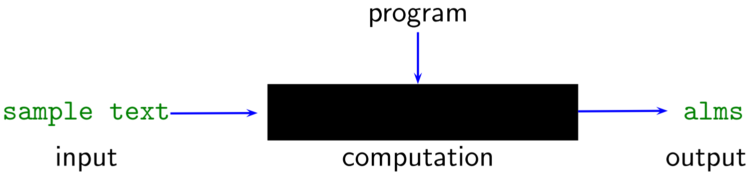
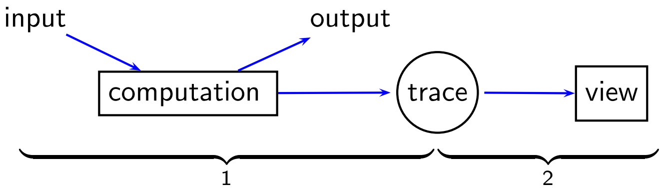
```
insert :: Ord a => a -> [a] -> [a]

insert x [] = [x]
insert x (y:ys) =
  if x > y then y : insert x ys
           else x : ys

sort :: Ord a => [a] -> [a]
sort [] = []
sort (x:xs) = insert x (sort xs)

main = getLine >>= putStrLn . sort
```



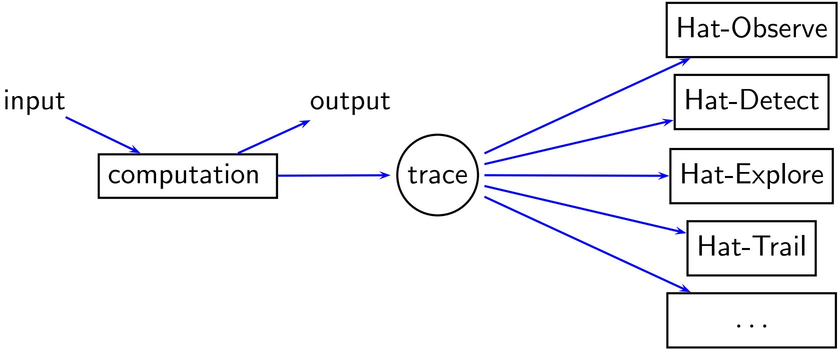
- Locate a fault (wrong output, run-time error, non-termination).
- Comprehend a program.

Two-Phase Tracing: A Trace as Data Structure



- Liberates from time arrow of computation.
- Enables views based on different execution models.
  (small-step, big-step, interpreter with environment, denotational)
- Enables compositional views.

- Multi-View Tracer



- Trace = Augmented Redex Trail (ART); distilled as unified trace.

Aim: A theoretical model of this trace and its views.

Overview

1. Definition of the Trace through Graph Rewriting
2. Properties of the Trace
3. Views of the Trace
   - Observation of Functions
   - Following Redex Trails
   - Algorithmic Debugging
4. Correctness of Algorithmic Debugging
5. Future Work & Summary

The Programming Language

Launchbury's and related semantics

- Subset of λ-calculus plus `case` for matching.
- Any program can be translated into this core calculus.

For tracing

- Close relationship between trace and original program essential.
- Language must have most frequently used features:
  - named functions
  - pattern matching

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For tracing

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- Language must have most frequently used features:
  - named functions
  - pattern matching

$\Rightarrow$ Higher-order term rewriting system

```
sort [] = []                          or   sort = foldr insert []
sort (x:xs) = insert x (sort xs)

insert x [] = [x]
insert x (y:ys) = if x > y then y : (insert x ys) else x : ys
```

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Program + input determine every detail of computation.

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What is a computation? Semantics answers:

- Term rewriting: A sequence of expressions.
  $t_1 \longrightarrow t_2 \longrightarrow t_3 \longrightarrow t_4 \longrightarrow t_5 \longrightarrow \ldots \longrightarrow t_n$
- Natural semantics: A proof tree.

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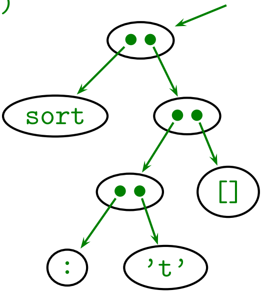
But

- Lots of redundancy.
- Much structure already lost.

Graph Rewriting I

`sort ('t':[])`

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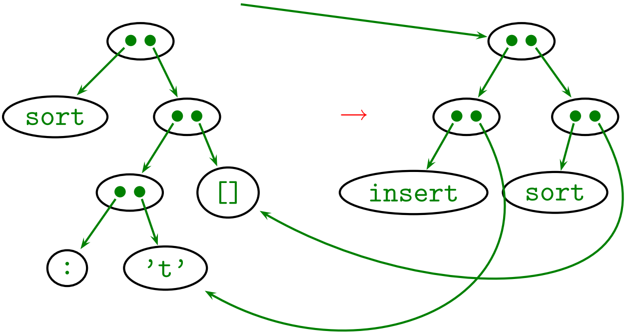


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- Create new nodes for right-hand-side.
- Nodes of subexpressions are shared.

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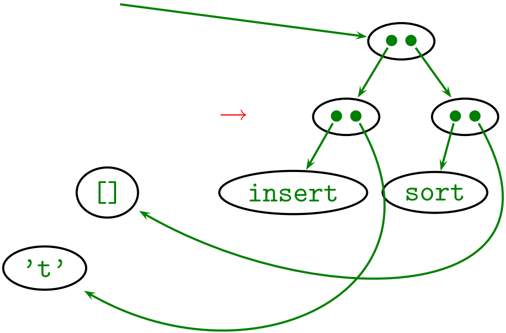


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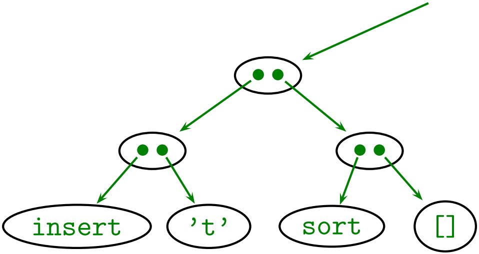
- Create new nodes for right-hand-side.
- Nodes of subexpressions are shared.
- Some old nodes become garbage.

Graph Rewriting II

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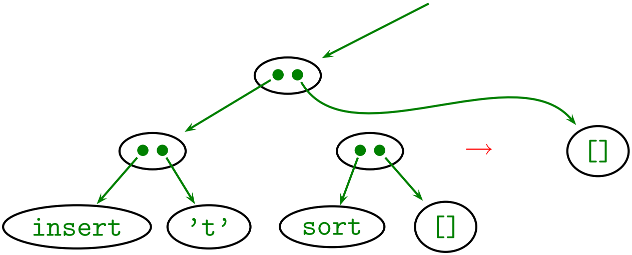


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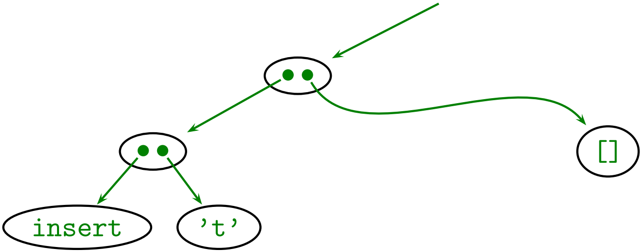
Graph Rewriting II

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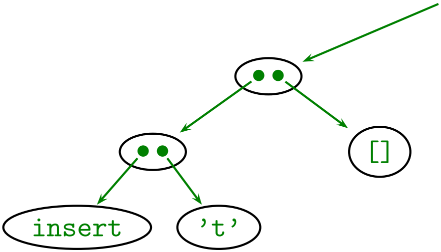
Graph Rewriting III

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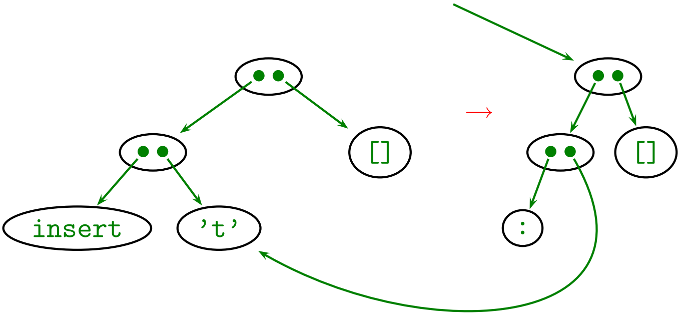


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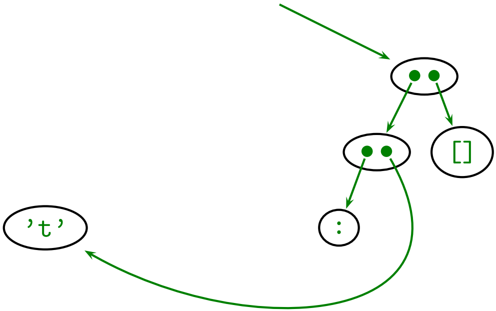


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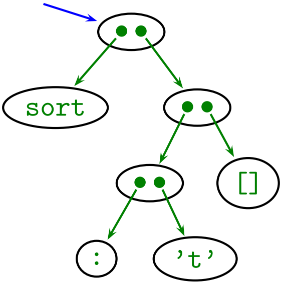
The Trace



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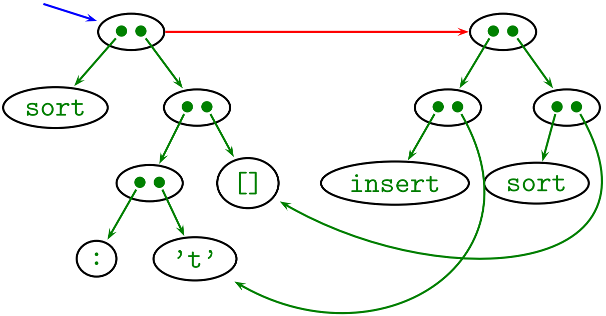

- New nodes for right-hand-side, connected via result pointer.
- Only add to graph, never remove.
- Sharing ensures compact representation.

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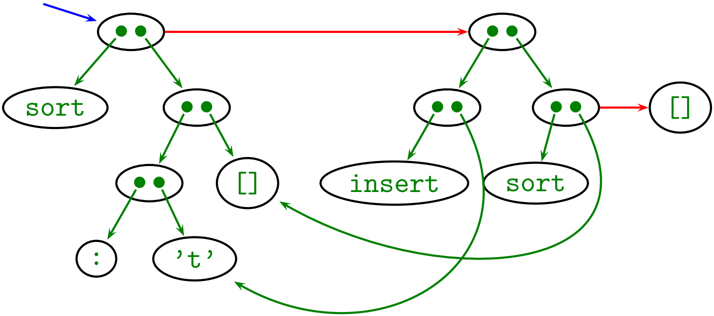

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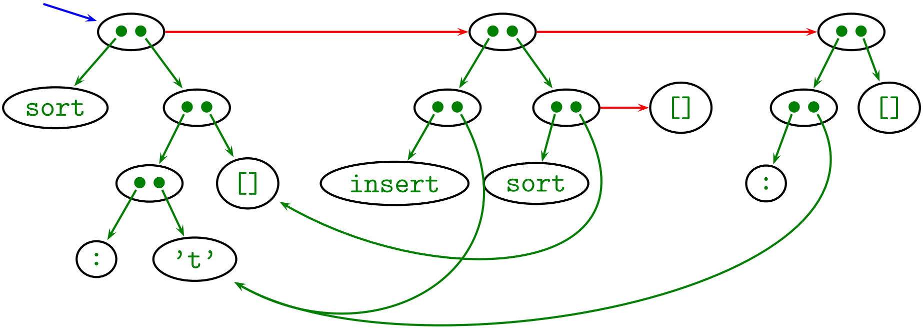

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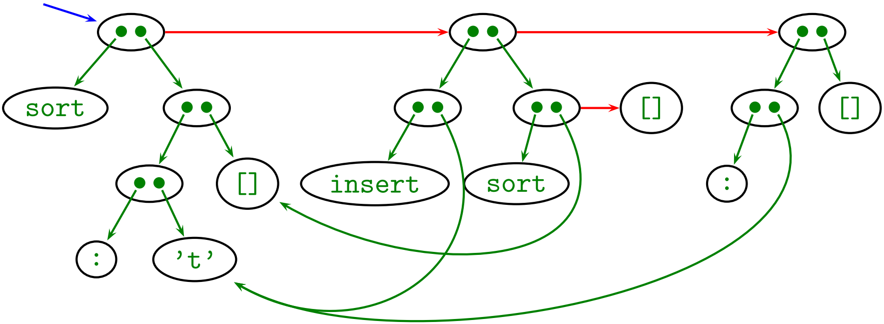
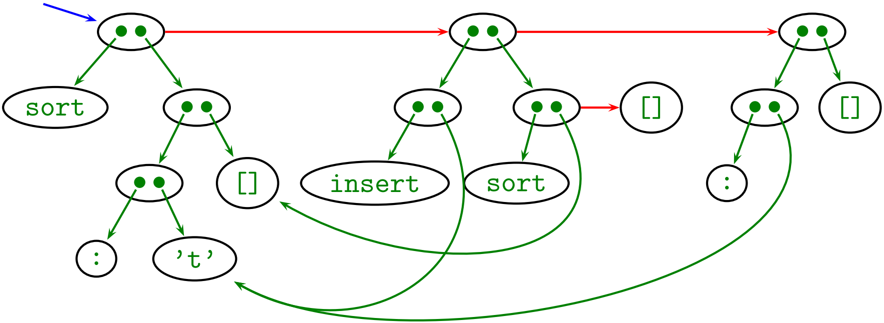
The Node Labels



term constructor	T	$:=$	a	atom
		$\vert$	$n\ m$	application of nodes
atom	a	$:=$	$f \mid C \mid 42 \mid \ldots$	defined variable, data constructor atomic literal, ...

- pointers instead of edges

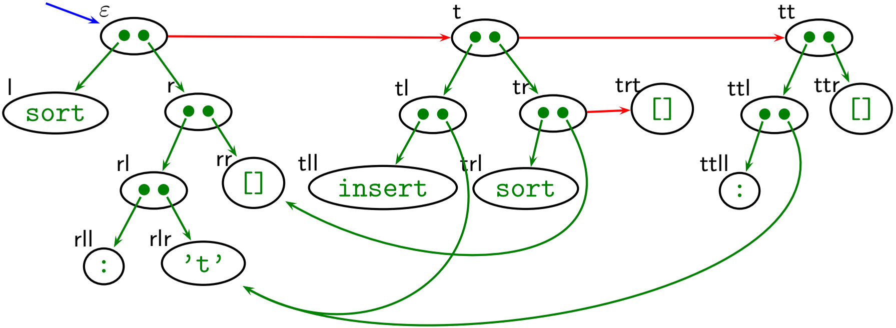
The Node Naming Scheme



Aim

- not distinguish isomorphic graphs
- avoid inconvenience of isomorphism classes

The Node Naming Scheme



Aim

- not distinguish isomorphic graphs
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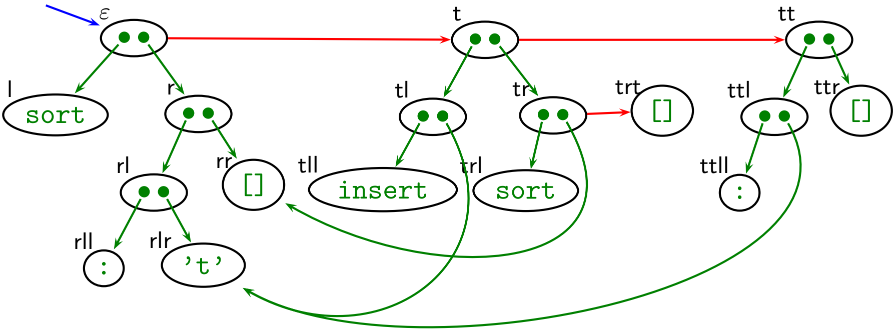
Solution

- standard representation with node describing path from root
- path at creation time (sharing later)
- path independent of evaluation order

The Node Naming Scheme II



- Reduction edge implicitly given through existence of node.
- Node encodes parent = top node of redex causing its creation:

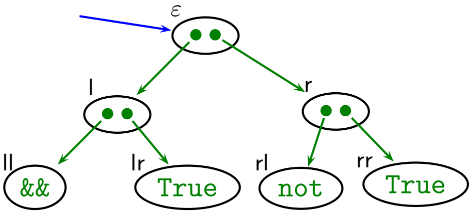
$$\begin{aligned} \mathsf{parent}(n\mathsf{t}) &= n \\ \mathsf{parent}(n\mathsf{l}) &= \mathsf{parent}(n) \\ \mathsf{parent}(n\mathsf{r}) &= \mathsf{parent}(n) \\ \mathsf{parent}(\varepsilon) &= \mathsf{undefined} \end{aligned}$$

- Easy to identify right-hand-side of rule: same parent.

Projections

- Reduction edge implicitly given through existence of node.
- Every redex should be parent of at least one node.
  (otherwise reduction unreachable from computation result)

```
True && x = x
not True = False
```



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$\Rightarrow$ A projection requires an indirection as result.

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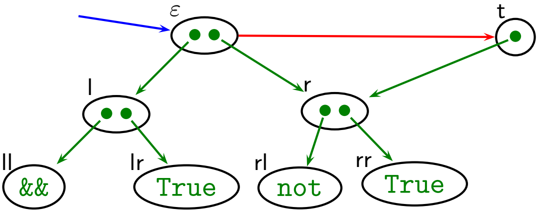

term constructor	T	$:=$	a	atom
		$\mid$	$n\ m$	application of nodes
		$\mid$	n	indirection

atom	a	$:=$	$x \mid C \mid 42 \mid \ldots$	variable, data constructor, ...

Projections

- Reduction edge implicitly given through existence of node.
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⇒ A projection requires an indirection as result.

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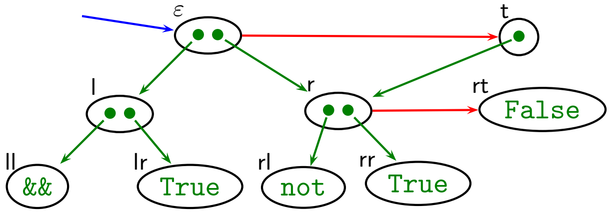

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The Trace: The Augmented Redex Trail (ART)

A trace G for initial term M and program P is a partial function from nodes to term constructors, $\mathcal{G} : n \mapsto T$, defined by

- The unshared graph representation of M, $\mathrm{graph}(\varepsilon, M)$, is a trace.
- If $\mathcal{G}$ is a trace and
  - $L = R$ an equation of the program P,
  - σ a substitution replacing argument variables by nodes,
  - $\mathrm{match}_{\mathcal{G}}(n, L\sigma)$,
  - $n\mathrm{t} \notin \mathrm{dom}(\mathcal{G})$,

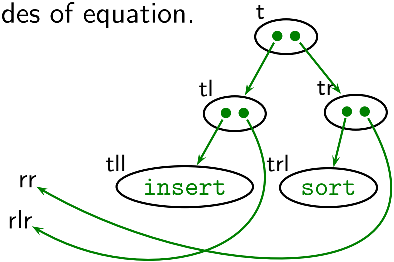
  then $\mathcal{G} \cup \mathrm{graph}(n\mathrm{t}, R\sigma)$ is a trace.

No evaluation order is fixed.

Unshared Graph Representation

For the initial term and right-hand-sides of equation.

$$\mathrm{graph}(\mathsf{t}, \mathtt{insert}\ \mathsf{rlr}\ (\mathsf{sort}\ \mathsf{rr})) =$$



Definition

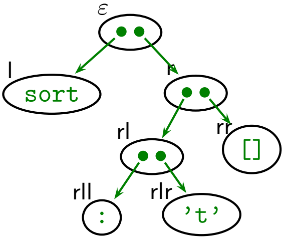
$$\mathrm{graph}(n, a) = \{(n, a)\}$$

$$\mathrm{graph}(n, m) = \{(n, m)\}$$

$$\mathrm{graph}(n, M\ N) = \begin{cases} \{(n, M\ N)\} & \text{, if } M\text{, } N \text{ are nodes} \\ \{(n, M\ n\mathsf{r})\} \cup \mathrm{graph}(n\mathsf{r}, N) & \text{, if only } M \text{ is a node} \\ \{(n, n\mathsf{l}\ N)\} \cup \mathrm{graph}(n\mathsf{l}, M) & \text{, if only } N \text{ is a node} \\ \{(n, n\mathsf{l}\ n\mathsf{r})\} \cup \mathrm{graph}(n\mathsf{l}, M) \cup \mathrm{graph}(n\mathsf{r}, N) & \text{, otherwise} \end{cases}$$

Matching

Matching a node with an instance of the left-hand-side of an equation.



$\mathrm{match}_{\mathcal{G}}(\varepsilon, \mathtt{sort}\ (\mathsf{rlr}\mathtt{:}\mathsf{rr}))$

Definition

$$\lceil n \rceil_{\mathcal{G}} = \text{if } n\mathrm{t} \in \mathrm{dom}(\mathcal{G}) \text{ then } \lceil n\mathrm{t} \rceil_{\mathcal{G}} \text{ else if } \exists m.\mathcal{G}(n) = m \text{ then } \lceil m \rceil_{\mathcal{G}} \text{ else } n$$

$$\begin{aligned}
\mathrm{match}_{\mathcal{G}}(o, a) &= (\mathcal{G}(o) = a) \\
\mathrm{match}_{\mathcal{G}}(o, M\, N) &= \exists m, n.(\mathcal{G}(o) = m\, n) \wedge \\
&\qquad ((m = M) \vee \mathrm{match}_{\mathcal{G}}(\lceil m \rceil_{\mathcal{G}}, M)) \wedge \\
&\qquad ((n = N) \vee \mathrm{match}_{\mathcal{G}}(\lceil n \rceil_{\mathcal{G}}, N)) \\
\mathrm{match}_{\mathcal{G}}(o, m) &= \mathrm{false}
\end{aligned}$$

Matching: Alternative Definitions

Matching a node with an instance of the left-hand-side of an equation.

Definition

$$
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\mathrm{match}_{\mathcal{G}}(o, a) &= (\mathcal{G}(o) = a) \\
\mathrm{match}_{\mathcal{G}}(o, M\,N) &= \exists m, n.(\mathcal{G}(o) = m\,n) \wedge \\
&\qquad (\text{if } M \text{ is a node then } (m = M) \text{ else } \mathrm{match}_{\mathcal{G}}(\lceil m \rceil_{\mathcal{G}}, M)) \wedge \\
&\qquad (\text{if } N \text{ is a node then } (n = N) \text{ else } \mathrm{match}_{\mathcal{G}}(\lceil n \rceil_{\mathcal{G}}, N)) \\
\mathrm{match}_{\mathcal{G}}(o, m) &= (o = m)
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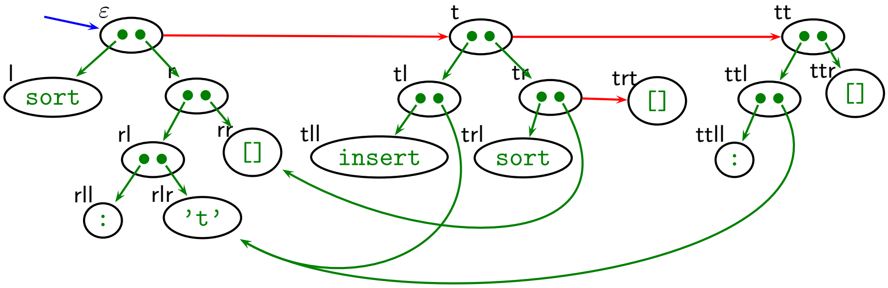
The Most Evaluated Form of a Node

A node represents many terms, in particular a most evaluated one.



$\mathsf{mef}_{\mathcal{G}}(\mathsf{tr}) = \texttt{[]}$

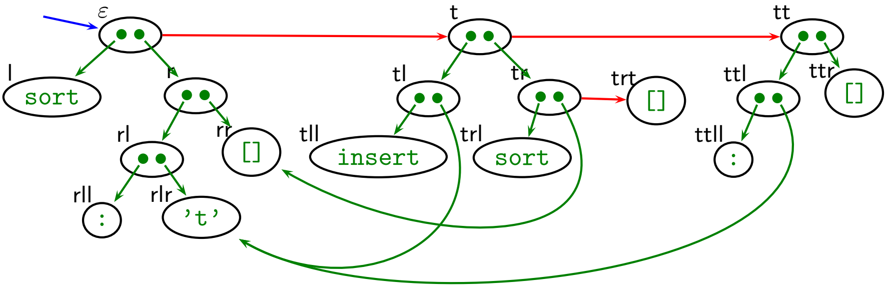
$\mathsf{mef}_{\mathcal{G}}(\varepsilon) = \texttt{(:)} \quad \texttt{'t'} \quad \texttt{[]}$

Definition

$$\mathsf{mef}_{\mathcal{G}}(n) = \mathsf{mefT}_{\mathcal{G}}(\mathcal{G}(\lceil n \rceil_{\mathcal{G}}))$$

$$\mathsf{mefT}_{\mathcal{G}}(a) = a$$

$$\mathsf{mefT}_{\mathcal{G}}(n\, m) = \mathsf{mef}_{\mathcal{G}}(n)\, \mathsf{mef}_{\mathcal{G}}(m)$$



$$\mathrm{redex}_{\mathcal{G}}(\mathrm{t}) = \texttt{insert 't' []}$$

$$\mathrm{bigstep}_{\mathcal{G}}(\mathrm{t}) = \texttt{insert 't' []} = \texttt{(:) 't' []}$$

Definition

For any redex node n, i.e., $nt \in \mathrm{dom}(\mathcal{G})$

$$\mathrm{redex}_{\mathcal{G}}(n) = \begin{cases} \mathrm{mef}_{\mathcal{G}}(m)\,\mathrm{mef}_{\mathcal{G}}(o) & \text{, if } \mathcal{G}(n) = m\,o \\ a & \text{, if } \mathcal{G}(n) = a \end{cases}$$

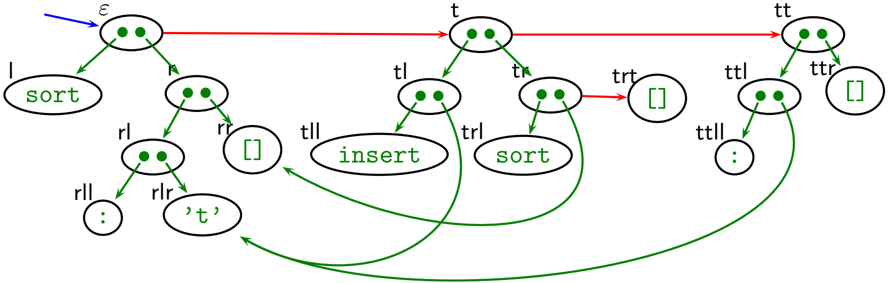
$$\mathrm{bigstep}_{\mathcal{G}}(n) = \mathrm{redex}_{\mathcal{G}}(n) = \mathrm{mef}_{\mathcal{G}}(n)$$

Properties of the ART

- closed (no dangling nodes)
- domain prefix-closed
- acyclic
- strongly confluent
- no application contains a node ending in t
- only a node ending in t can be an indirection
- if $n\mathsf{l} \in \mathsf{dom}(\mathcal{G})$, then $\mathcal{G}(n) = n\mathsf{l}\ m$
- if $n\mathsf{r} \in \mathsf{dom}(\mathcal{G})$, then $\mathcal{G}(n) = m\ n\mathsf{r}$
- if $n\mathsf{t} \in \mathsf{dom}(\mathcal{G})$, then $\mathsf{redex}_{\mathcal{G}}(n) = L\sigma$ and $\mathsf{reduct}_{\mathcal{G}}(n) = R\sigma$ for some program equation $L = R$ and substitution σ

Give non-inductive definition of ART based on properties?

Reduct of a Small Step Reduction



$\mathit{reduct}_{\mathcal{G}}(\varepsilon) =$ `insert 't' (sort [])`

Definition

$$\mathrm{reduct}_{\mathcal{G}}(n) = \mathrm{reductP}_{\mathcal{G}}(n, n\mathrm{t})$$

$$\mathrm{reductP}_{\mathcal{G}}(p, n) = \text{if } \mathrm{parent}(n) = p \text{ then } \mathrm{reductT}_{\mathcal{G}}(p, \mathcal{G}(n)) \text{ else } \mathrm{mef}_{\mathcal{G}}(n)$$

$$\mathrm{reductT}_{\mathcal{G}}(p, a) = a$$

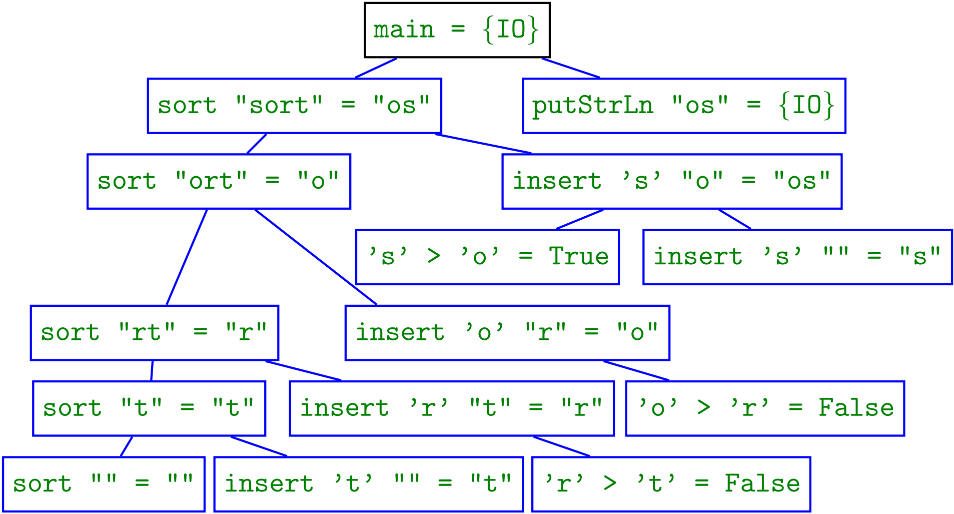
$$\mathrm{reductT}_{\mathcal{G}}(p, m) = \mathrm{mef}_{\mathcal{G}}(m)$$

$$\mathrm{reductT}_{\mathcal{G}}(p, n\, o) = \mathrm{reductP}_{\mathcal{G}}(p, n)\, \mathrm{reductP}_{\mathcal{G}}(p, o)$$

Algorithmic Debugging

```
sort "sort" = "os"?   n

insert 's' "o" = "os"?   y

sort "ort" = "o"?   n

insert 'o' "r" = "o"?   n

Bug identified:
  "Insert.hs":8-9:
  insert x [] = [x]
  insert x (y:ys) = if x > y then y:(insert x ys) else x:ys
```

The Evaluation Dependency Tree



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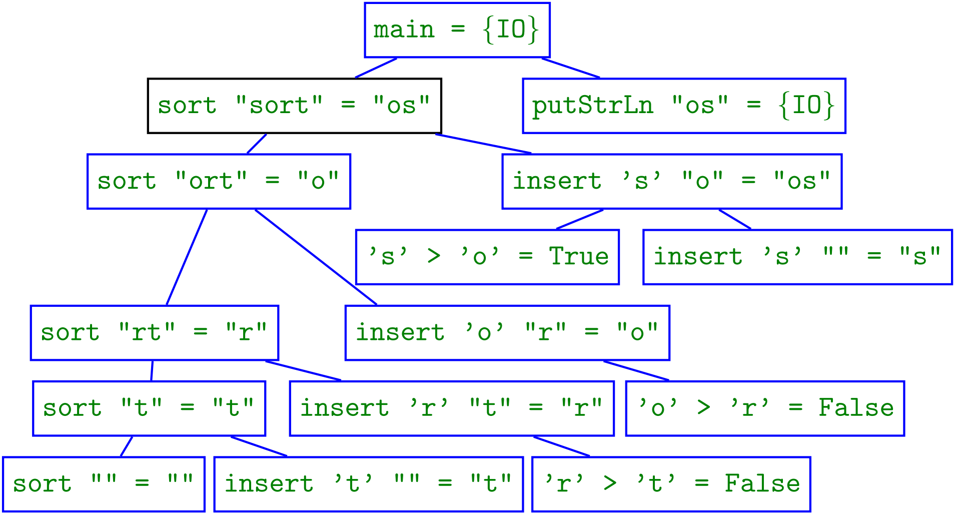

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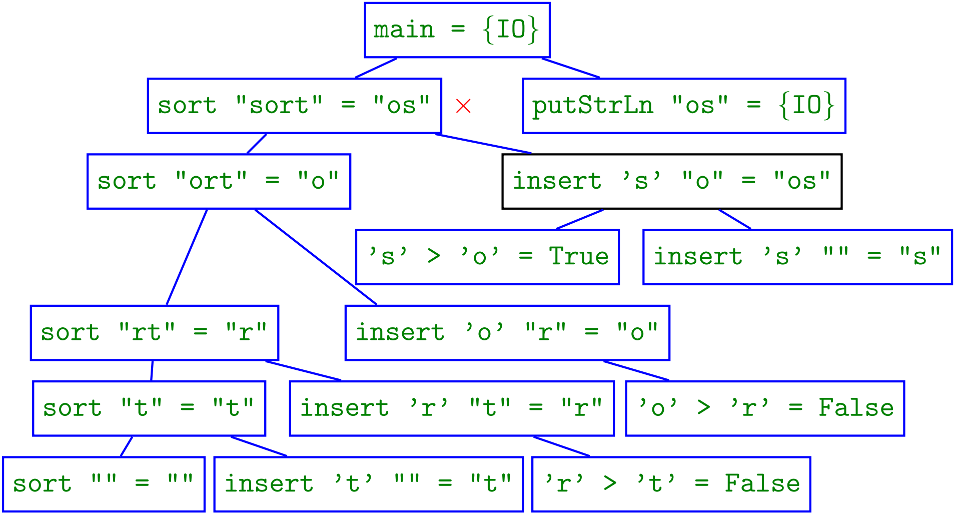

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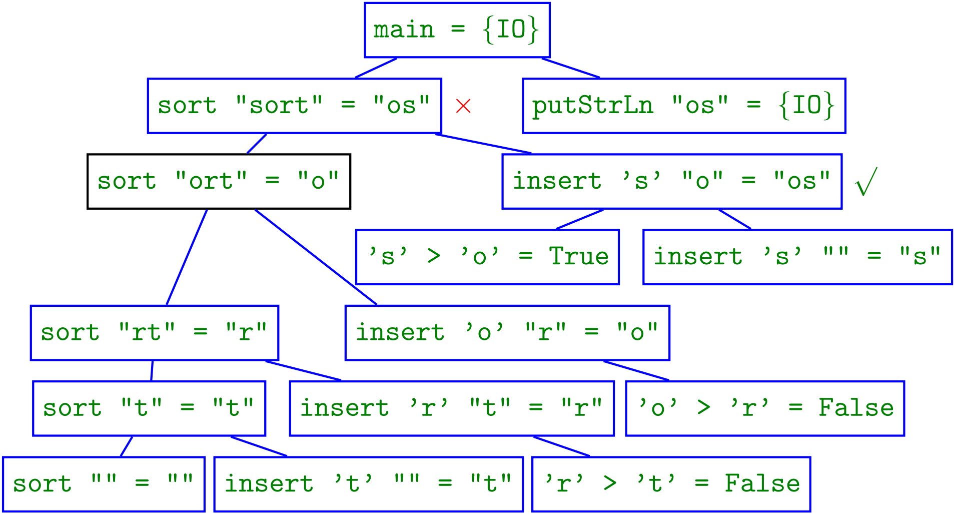

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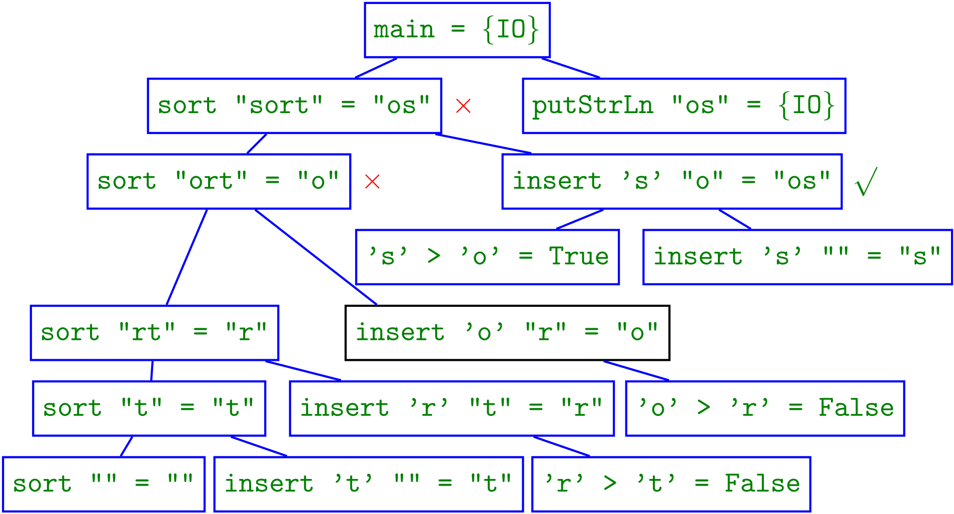

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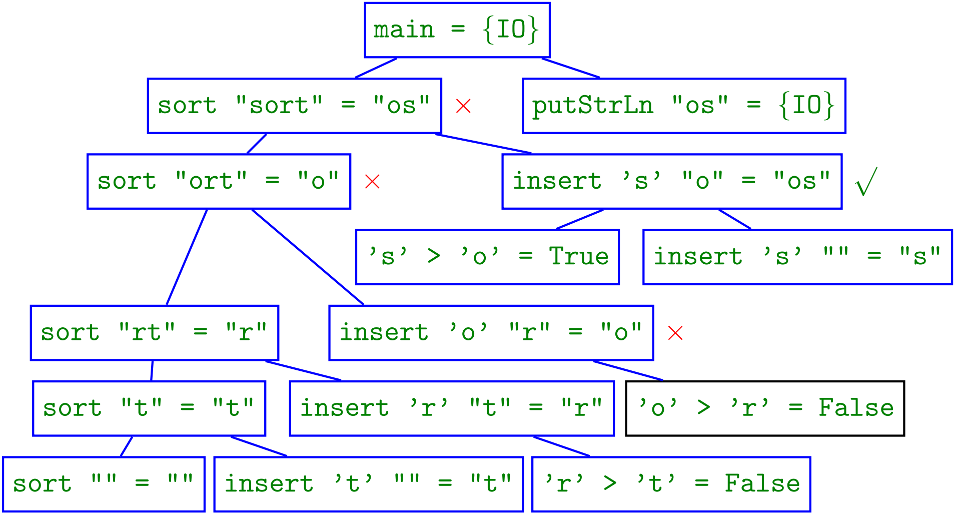
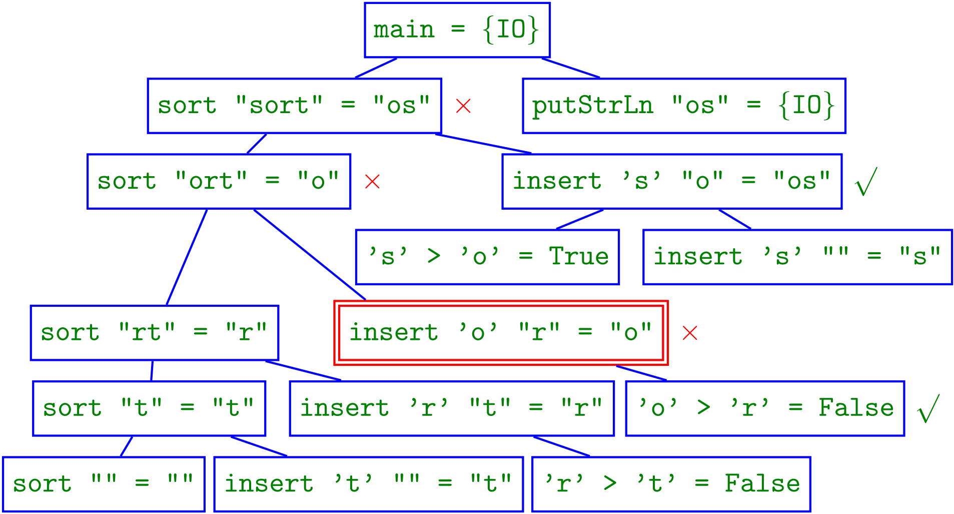
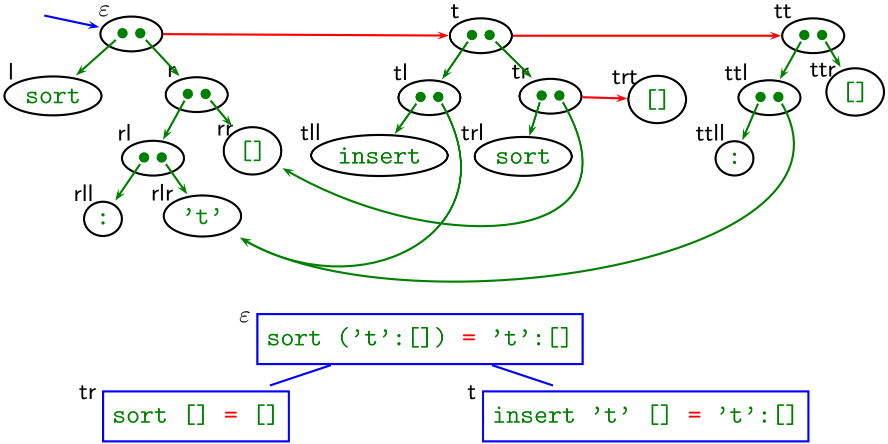


The Evaluation Dependency Tree



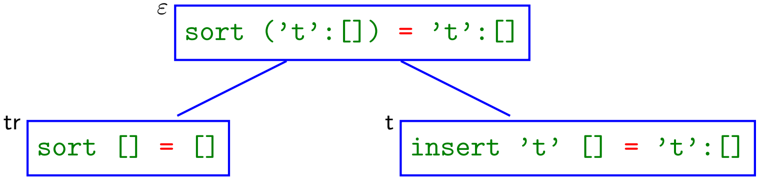
The ART and the Evaluation Dependency Tree



- Every redex node n yields a tree node n labelled $\text{bigstep}_{\mathcal{G}}(n)$.
- Tree node n is child of tree node $\text{parent}(n)$.
- Usually root label $\text{bigstep}_{\mathcal{G}}(\varepsilon) = \texttt{main = ...}$

Correctness of Algorithmic Debugging: The Property

If node n incorrect and all its children correct, then node n faulty, i.e., its equation is faulty.



Definition

Tree node n incorrect $\Leftrightarrow$ $\text{redex}_{\mathcal{G}}(n) \not\cong_{\mathsf{I}} \text{mef}_{\mathcal{G}}(n)$.

Tree node n faulty $\Leftrightarrow$ $\text{redex}_{\mathcal{G}}(n) \not\cong_{\mathsf{I}} \text{reduct}_{\mathcal{G}}(n)$.

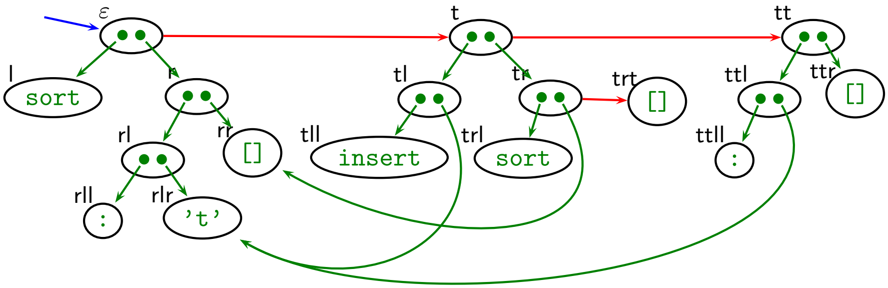
If tree node n faulty, then for its program equation $L = R$ exists substitution σ such that $L\sigma \not\cong_{\mathsf{I}} R\sigma$.

Correctness of Algorithmic Debugging: Main Theorem

Theorem

Let n be a redex node. If for all redex nodes m with $parent(m) = n$ we have $redex_{\mathcal{G}}(m) \cong_I mef_{\mathcal{G}}(m)$, then $reduct_{\mathcal{G}}(n) \cong_I mef_{\mathcal{G}}(n)$.

With $\mathrm{redex}_{\mathcal{G}}(n) \ncong_I \mathrm{mef}_{\mathcal{G}}(n)$ follows $\mathrm{redex}_{\mathcal{G}}(n) \ncong_I \mathrm{reduct}_{\mathcal{G}}(n)$.

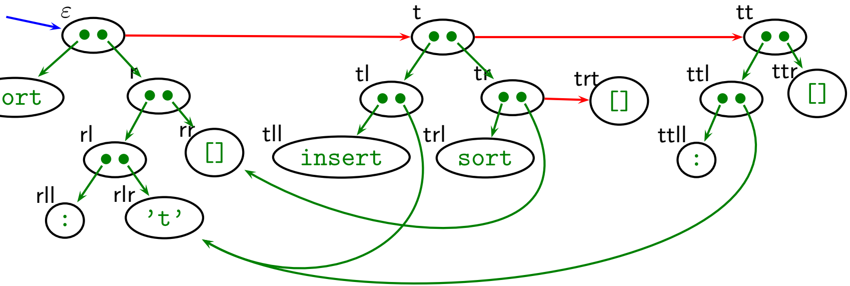


Correctness of Algorithmic Debugging: Proof

Proof.

Generalise property: Let $n \in \mathrm{dom}(\mathcal{G})$. If for all redex nodes m with $\mathrm{parent}(m) = \mathrm{parent}(n)$ we have $\mathrm{redex}_{\mathcal{G}}(m) \cong_{\mathsf{I}} \mathrm{mef}_{\mathcal{G}}(m)$, then $\mathrm{reduct}_{\mathcal{G}}(n) \cong_{\mathsf{I}} \mathrm{mef}_{\mathcal{G}}(n)$.

Induction over $\mathrm{hight}_{\mathcal{G}}(n) = \max\{|o| \mid o \in \{\mathsf{l},\mathsf{r}\}^* \wedge no \in \mathrm{dom}(\mathcal{G})\}$. □



Conclusions

- Simple model amenable to proof.
- Contains a wealth of information about computation.
- Models real-world trace of Haskell tracer Hat.