Modularity of convergence in infinitary rewriting

Stefan Kahrs

University of Kent at Canterbury

Overview

- a brief recap on infinitary rewriting and convergence
- the problem
- some examples
- metric abstract reduction systems
- sketch the proof

Recap: Infinitary Term Rewriting

- in infinitary rewriting we permit infinite terms, and have transfinite reductions approximating them
- reduction sequence: a continuous function $f:\alpha \to Ter^{\infty}(\Sigma)$ such that $f(n) \to f(n+1)$
- open reduction sequence f: α is a limit ordinal; it is converging if we can extend the domain to $\alpha+1$, keeping it continuous
- f is strongly convergent if in addition redex positions are eventually deep

The problem

- an iTRS is (strongly) convergent iff all its open reduction sequences are
- are these modular properties of iTRSs?
- in general, or under certain conditions?

Example 1: collapsing rules

- $F(x) \to x$; $G(y) \to y$
- each rule on its own convergent, but not together
- $t = F(G(t))$, $u = G(F(u))$
- $t \to u \to t \to \ldots$
- note: the presence of a collapsing rule always breaks strong convergence

Example 2: one collapsing rule

- $F(x) \to x$; $G(H(x)) \to G(x)$
- each rule on its own convergent, but not together
- $t = F(H(t))$, $u = H(F(u))$
- $G(t) \to G(u) \to G(t) \to \ldots$

Revised Problem

- is convergence modular for non-collapsing iTRS?

Example 3: weird stuff

- $F(x,x,y)\to F(x,y,x)$; $0\to S(0)$
- both are individually convergent, and they are together as well; but notice:
- $F(0,0,0)\to F(0,0,1)\to F(0,1,0)\to F(1,1,0)\to F(1,0,1)\to F(1,1,1)\to \ldots$
- if we project the second argument we get the sequence
- 0, 0, 1, 1, 0, 1, 1, 2, 2, 1, 2, 2, 3, 3, 2,...
- not a reduction sequence!

Example 4: more weird stuff

- $A \to H(A)$, $A \to Z$, $H(Z) \to S(Z)$, $H(S(x)) \to S(S(x))$
- we have $A \to_w S^n(Z)$ but we do not have $A \to_w S^\infty$
- $J(K(x,y)) \to J(y)$
- $t=K(A,t)$, $u=K(S^\infty,u)$ we have $J(t) \to_w J(u)$
- ...but the blue subterms of J(t) do not reduce to the blue subterms of J(u)

Metric abstract reduction systems

- a MARS is an ARS with a metric $(M,\to,d)$
- sequences: ordinal-indexed continuous functions
- weak reduction sequences: reduction sequences of the MARS $(M,\to_w,d)$
- theorem: $(M,\to,d)$ is convergent iff $(M,\to_w,d)$ is

Focussed Sequences

- a sequence $f:\alpha\to M$ is focussed, if there is a $\beta<\alpha$, such that for all $\gamma\geq\beta$, there is a ζ:
- $f(\gamma)\to_w f(\kappa)$ for all κ, $\alpha>\kappa\geq\zeta$
- in words: elements sufficiently far down the sequence reduce to all elements sufficiently far down the sequence
- theorem: a MARS is convergent iff all its focussed sequences are

Replacing principal subterms

- write t[n↘u] for replacing all principal subterms, n ranks from the root, by the fixed term u
- this operation preserves (reflexive) reduction steps
- we also use it for sequences, applying it pointwise

Set up

- in the following let R and S be two non-collapsing and convergent iTRSs
- let f be an open reduction sequence of the combined system
- observe that f[1↘x] must converge, by assumption
- corollary: strong convergence is modular

Proof idea

- if $l \rightarrow r$ in the system at the root (and $l \neq r$) of the sequence then f[1↘l] must be convergent
- it must remain convergent if we reduce some of the l's to r's
- this tells us something about f

Predicate sequence

- a predicate sequence is a function that tells us whether we should replace a term (a principal subterm) with l or r, depending on how far we are in the sequence
- we need to ensure that reductions are preserved:
  - once we replace t with r, we need to do this as well further down the sequence
  - if we replace t with r, and $t \rightarrow_w s$ then we replace s with r as well

Modified Sequence

- each rewrite step of the original sequence is split into two halves
- first half: time is moved on, and so some l's will be rewritten to r's; this also models rewriting below principal subterm positions
- second half: the original rewrite step is performed (if situated in top-rank)

2 particular ones

- k_t: a term is not a reduct of t
- f_p: a reduct of the term will appear in position p "in the future"

Observations

- by modifying f with predicate sequence f_p we can show that all principal subterms in position p sufficiently far down the sequence have a reduct further down
- by modifying it with k_t one can show that the sequence of subterms at p is focussed

Overall proof

- if f is divergent then it is divergent with diameter ε
- thus it suffices to look at f[n↘x] with $2^{(-n)} < \varepsilon$
- then we can prove the property by induction on the rank, using the previous observation, and the earlier theorem that convergence of focussed sequences coincides with convergence of reduction sequences

Conclusion

- convergence is modular for non-collapsing iTRSs
- strong convergence is modular

  (note: for left-linear systems this result is due to Simonsen)