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New eta-reduction and Church-Rosser

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Abstract This paper introduces a new eta-reduction rule for λ-calculus with dependent types and prove the property of Church-Rosser.

1 Introduction

For the simple λ-calculus, the property of Church-Rosser w.r.t. β and η-reduction holds, no matter a term is well-typed or not, and the methods of proving this property vary [Bar84,Tak95,Nip01]. Some systems such as Edinburgh Logical Framework [HHP87,HHP92] have β-reduction only, and the proof method is known as that of Tait and Martin-Löf. However, for some dependently typed systems such as Martin-Löf's Logical Framework [ML84,NPS90,Luo94], the story becomes very different because ill-typed terms may not have this property w.r.t. β and η-reduction. The proofs for such systems are very difficult because one has to prove that "well-typed terms have the property of Church-Rosser" [Gog94]. In this paper, a new eta-reduction is introduced in order to give an easy proof of the property of Church-Rosser, and the property does not rely on whether terms are well-typed or not. The new eta-reduction is sufficient for our understanding of eta-reduction in dependent type systems, in the sense that a property that holds w.r.t. the old one also holds w.r.t. the new one.

In Section 2, we give counter examples to demonstrate the problem with the property of Church-Rosser w.r.t. the β-reduction and the old η-reduction. In Section 3, new eta-reduction η' and its one-step reduction are introduced, and the relations between the new and the old (normal) eta-reduction are discussed. In Section 4, we prove the property of Church-Rosser w.r.t. β and η'-reduction, which holds without the condition of well-typedness. The conclusion is given in the last section.

2 Old reduction rules

In this section, we present a basic definition of terms and knids, and the old β and eta-reduction rules. Counter examples are also given to demonstrate the problem with the property of Church-Rosser.

Definition 1. ***(Terms and Kinds)***

- *Term*
  1. *a variable is a term,*
  2. *$\lambda x : K.M$ is a term if x is a variable, K is a kind and M is a term,*
  3. *MN is a term if M and N are terms.*
- *Kind*
  1. *Type is a kind,*
  2. *$El(M)$ is a kind if M is a term,*
  3. *$(x : K_1)K_2$ is a kind if K_1 and K_2 are kinds.*

Remark 1. Terms and kinds are mutually and recursively defined.

Notation: Following the tradition, Λ denotes the set of all terms and Π the set of all kinds. We shall use M, N, P, Q, R for arbitrary terms, and K for an arbitrary kind, and x, y, z for arbitrary variables. We also write A for $El(A)$ when no confusion may occur.

The old (normal) reduction rules:

$$(\lambda x : K.M)N \longrightarrow_\beta [N/x]M$$

$$\lambda x : K.Mx \longrightarrow_\eta M \qquad x \notin FV(M)$$

Example 1. **(Counter examples)** The property of Church-Rosser may not hold for ill-typed terms. The following examples will show the problem.

$$\lambda x : A.(\lambda y : B.y)x \longrightarrow_\beta \lambda x : A.x$$

$$\lambda x : A.(\lambda y : B.y)x \longrightarrow_\eta \lambda y : B.y$$

$$\begin{aligned} & (\lambda z : (y : B)B.\lambda x : A.zx)(\lambda y : B.y) \\ \longrightarrow_\beta\ & \lambda x : A.(\lambda y : B.y)x \\ \longrightarrow_\beta\ & \lambda x : A.x \end{aligned}$$

$$\begin{aligned} & (\lambda z : (y : B)B.\lambda x : A.zx)(\lambda y : B.y) \\ \longrightarrow_\eta\ & (\lambda z : (y : B)B.z)(\lambda y : B.y) \\ \longrightarrow_\beta\ & \lambda y : B.y \end{aligned}$$

where A and B are distinct variables. There is no common reduct for $\lambda x : A.x$ and $\lambda y : B.y$.

3 New eta-reduction rule

In this section, we introduce a new eta-reduction rule and its one-step reduction, and discuss the relations between the old and new.

Definition 2. ***(Base terms)***

- *a variable is a base term,*
- *MN is a base term if M is a base term.*

Notation: We shall use $\overline{R}$ for $R_1, R_2, ..., R_n$ for some $n \geq 0$, and $M\overline{R}$ for $(...((MR_1)R_2)...R_n)$, and $x\overline{R}$ for an arbitrary base term.

Definition 3. ***(Head of terms)***

- *$head(x) = x$,*
- *$head(\lambda x : K.M) = \lambda x : K.M$,*
- *$head(MN) = head(M)$.*

Lemma 1. *If a term is not a base term, then the head of the term is of the form $\lambda x : K.M$.*

New redex for eta-reduction

There are two different forms of redexes: $(\lambda x : K.M)N$ and $\lambda x : K.(y\overline{N})x$ when $x \notin FV(y\overline{N})$. The reduction rules for these redexes are the following.

$$(\lambda x : K.M)N \longrightarrow_\beta [N/x]M$$

$$\lambda x : K.(y\overline{N})x \longrightarrow_{\eta'} y\overline{N} \qquad x \notin FV(y\overline{N})$$

The one-step β and η'-reduction are defined in Figure 1.

Remark 2. We have the following remarks.

- The one-step β-reduction is the same as usual, but the redex for the new eta-reduction rule (η') has a more restricted form (*i.e.* $y\overline{N}$ is a base term). For the term $\lambda x : A.(\lambda y : B.y)x$ in Example 1, the new eta-reduction rule (η') cannot be applied because $\lambda y : B.y$ is not a base term.
- It is not the case that one has to do all the β-reductions first, then do η'-reductions when reducing a term. For example, we have

$$x(\lambda y : B.zy)((\lambda x : A.x)a) \longrightarrow_{\eta'} xz((\lambda x : A.x)a) \longrightarrow_\beta xza$$

One-step β-reduction:

$$\frac{}{(\lambda x:K.M)N \longrightarrow_\beta [N/x]M}$$

$$\frac{M \longrightarrow_\beta M'}{MN \longrightarrow_\beta M'N} \qquad \frac{N \longrightarrow_\beta N'}{MN \longrightarrow_\beta MN'}$$

$$\frac{M \longrightarrow_\beta M'}{\lambda x:K.M \longrightarrow_\beta \lambda x:K.M'} \qquad \frac{K \longrightarrow_\beta K'}{\lambda x:K.M \longrightarrow_\beta \lambda x:K'.M}$$

$$\frac{M \longrightarrow_\beta M'}{El(M) \longrightarrow_\beta El(M')}$$

$$\frac{K_1 \longrightarrow_\beta K_1'}{(x:K_1)K_2 \longrightarrow_\beta (x:K_1')K_2} \qquad \frac{K_2 \longrightarrow_\beta K_2'}{(x:K_1)K_2 \longrightarrow_\beta (x:K_1)K_2'}$$

One-step η'-reduction:

$$\frac{x \notin FV(y\overline{N})}{\lambda x:K.(y\overline{N})x \longrightarrow_{\eta'} y\overline{N}}$$

$$\frac{M_i \longrightarrow_{\eta'} M_i' \qquad n \geq 1}{xM_1...M_i...M_n \longrightarrow_{\eta'} xM_1...M_i'...M_n}$$

$$\frac{M \longrightarrow_{\eta'} M'}{\lambda x:K.M \longrightarrow_{\eta'} \lambda x:K.M'} \qquad \frac{K \longrightarrow_{\eta'} K'}{\lambda x:K.M \longrightarrow_{\eta'} \lambda x:K'.M}$$

$$\frac{M \longrightarrow_{\eta'} M'}{El(M) \longrightarrow_{\eta'} El(M')}$$

$$\frac{K_1 \longrightarrow_{\eta'} K_1'}{(x:K_1)K_2 \longrightarrow_{\eta'} (x:K_1')K_2} \qquad \frac{K_2 \longrightarrow_{\eta'} K_2'}{(x:K_1)K_2 \longrightarrow_{\eta'} (x:K_1)K_2'}$$

Figure 1. One-step reduction

Notations: Let $\longrightarrow_R$ be one-step R-reduction. We denote $\longrightarrow_R^=$ as the reflexive closure of $\longrightarrow_R$, and $\twoheadrightarrow_R$ as the the reflexive, transitive closure of $\longrightarrow_R$, and $=_R$ as the R-convertibility.

Lemma 2. *If $x \notin FV(M)$ and $M \twoheadrightarrow_{\beta\eta'} N$, then $x \notin FV(N)$.*

In some cases, although there is a η'-redex inside a term, this term may not have one-step η'-reduction. For instance, $(\lambda x : A.yx)z$ only has β-reduction although $\lambda x : A.yx$ is a η'-redex. However, we have the following lemma.

Lemma 3. *If there is a η'-redex inside M, then M has a one-step reduction (*i.e.* either β or η'-reduction).*

Proof. By induction on M.

- If M itself is a η'-redex, then the statement is true.
- If $M \equiv xN_1...N_n$ and the redex is inside N_i, then by induction hypothesis, N_i has either β or η'-reduction. By the one-step reduction rules in Figure 1, M has a one-step reduction.
- If $M \equiv \lambda x : K.N$ and the redex is inside K or N, then by induction hypothesis, K or N has either β or η'-reduction. By the one-step reduction rules, M has a one-step reduction.
- If $M \equiv (\lambda x : K.P)N_1...N_n$ $(n \geq 1)$, then M has at least a one-step β-reduction.
- Other cases for kinds are trivial.

Relations between the old and new

Theorem 1. *A normal form w.r.t. $\beta\eta'$ is also a normal form w.r.t. $\beta\eta$, and* vice versa.

Proof. We proceed the proof by contradiction. An interesting case is to prove that, a η-redex contains a β or η'-redex. Then by Lemma 3, the whole term has a one-step reduction.

Suppose $\lambda x : K.Mx$ is a η-redex, *i.e.* $x \notin FV(M)$. If M is a base term, then $\lambda x : K.Mx$ is a η'-redex. If M is not a base term, then by Lemma 1, the head of M must be of the form $\lambda x : K'.N$. So, Mx must contain a β-redex. □

Lemma 4. *A normal form (w.r.t. $\beta\eta$ or $\beta\eta'$) is either of the form $\lambda y : K'.N$ or of the form $y\overline{N}$ (a base term).*

Conjecture 1. If $x \notin FV(M)$ and $\lambda x : K.Mx$ is well-typed, then $\lambda x : K.Mx =_{\beta\eta'} M$.

Proof. We only give an informal proof here since the well-typedness has not been defined yet. We are going to prove that the property of Church-Rosser w.r.t. $\beta\eta'$-reduction holds for any term, no matter it is well-typed or not. So well-typedness is out of scope of the paper, but the typing inference rules can be found in Appendix. We assume that well-typed terms have good properties such as strong normalisation.

Since $\lambda x : K.Mx$ is well-typed, it is strongly normalising, and so is M. We first reduce M to a normal form M'. By Lemma 4, M' is either of the form $\lambda y : K'.N$ or of the form $y\overline{N}$.

If $M' \equiv y\overline{N}$, then we have

$$\begin{aligned}
\lambda x : K.Mx &=_{\beta\eta'} \lambda x : K.M'x \\
&\equiv \lambda x : K.(y\overline{N})x \\
&=_{\eta'} y\overline{N} \\
&=_{\beta\eta'} M
\end{aligned}$$

Note that $x \notin FV(y\overline{N})$ by Lemma 2.

If $M' \equiv \lambda y : K'.N$, because $\lambda x : K.(\lambda y : K'.N)x$ is well-typed, we have $K =_{\beta\eta'} K'$. Then we have

$$\begin{aligned}
\lambda x : K.Mx &=_{\beta\eta'} \lambda x : K.M'x \\
&\equiv \lambda x : K.(\lambda y : K'.N)x \\
&=_{\beta\eta'} \lambda x : K'.(\lambda y : K'.N)x \\
&=_{\beta} \lambda x : K'.[x/y]N \\
&=_{\alpha} \lambda y : K'.N \\
&=_{\beta\eta'} M
\end{aligned}$$

□

4 Proof of Church-Rosser

In this section, we give some important definitions such as Parallel Reduction and Complete Development for β-reduction, and prove the property of Church-Rosser.

Definition 4. ***(Parallel reduction for β)*** *The parallel reduction, denoted by $\Rightarrow_\beta$, is defined inductively as follows.*

1. $x \Rightarrow_\beta x$,
2. $\lambda x : K.M \Rightarrow_\beta \lambda x : K'.M'$ *if* $K \Rightarrow_\beta K'$ *and* $M \Rightarrow_\beta M'$,
3. $MN \Rightarrow_\beta M'N'$ *if* $M \Rightarrow_\beta M'$ *and* $N \Rightarrow_\beta N'$,
4. $(\lambda x : K.M)N \Rightarrow_\beta [N'/x]M'$ *if* $M \Rightarrow_\beta M'$ *and* $N \Rightarrow_\beta N'$,
5. $Type \Rightarrow_\beta Type$,
6. $El(M) \Rightarrow_\beta El(M')$ *if* $M \Rightarrow_\beta M'$,
7. $(x : K_1)K_2 \Rightarrow_\beta (x : K_1')K_2'$ *if* $K_1 \Rightarrow_\beta K_1'$ *and* $K_2 \Rightarrow_\beta K_2'$.

Based on the inductive definition of $\Rightarrow_\beta$, we have the following lemma.

Lemma 5. *We have the following properties, where M and M' represent arbitrary terms or kinds, and N and N' are arbitrary terms.*

1. $M \Rightarrow_\beta M$.
2. *If* $M \longrightarrow_\beta M'$ *then* $M \Rightarrow_\beta M'$.
3. *If* $M \Rightarrow_\beta M'$ *then* $M \twoheadrightarrow_\beta M'$.
4. *If* $M \Rightarrow_\beta M'$ *and* $N \Rightarrow_\beta N'$ *then* $[N/x]M \Rightarrow_\beta [N'/x]M'$.

Proof. For (1), (3) and (4), by induction on M; for (2), by induction on the context of the redex. We only prove the most difficult case (4) here.

1. If $M \equiv x$, then $x \Rightarrow_\beta x \equiv M'$ and hence

$$[N/x]M \equiv N \Rightarrow_\beta N' \equiv [N'/x]M'$$

2. If $M \equiv y \not\equiv x$, then $y \Rightarrow_\beta y \equiv M'$ and hence

$$[N/x]M \equiv y \Rightarrow_\beta y \equiv [N'/x]M'$$

3. If $M \equiv \lambda y : K.P$, then $\lambda y : K.P \Rightarrow_\beta \lambda y : K'.P' \equiv M'$ and $K \Rightarrow_\beta K'$ and $P \Rightarrow_\beta P'$. By induction hypothesis, we have $[N/x]K \Rightarrow_\beta [N'/x]K'$ and $[N/x]P \Rightarrow_\beta [N'/x]P'$. Hence

$$[N/x]M \equiv \lambda y : [N/x]K.[N/x]P \Rightarrow_\beta [N'/x]M'$$

4. If $M \equiv PQ$, then there are two sub-cases.
   (a) $PQ \Rightarrow_\beta P'Q' \equiv M'$ and $P \Rightarrow_\beta P'$ and $Q \Rightarrow_\beta Q'$. By induction hypothesis, we have $[N/x]P \Rightarrow_\beta [N'/x]P'$ and $[N/x]Q \Rightarrow_\beta [N'/x]Q'$. Hence $[N/x]M \Rightarrow_\beta [N'/x]M'$.
   (b) $PQ \equiv (\lambda y : K.R)Q \Rightarrow_\beta [Q'/y]R' \equiv M'$ and $R \Rightarrow_\beta R'$ and $Q \Rightarrow_\beta Q'$. By induction hypothesis, we have $[N/x]R \Rightarrow_\beta [N'/x]R'$ and $[N/x]Q \Rightarrow_\beta [N'/x]Q'$. Hence

$$[N/x]M \Rightarrow_\beta [([N'/x]Q')/y]([N'/x]R') \equiv [N'/x]M'$$

5. If $M \equiv Type$, then $Type \Rightarrow_\beta Type \equiv M'$ and hence $[N/x]M \Rightarrow_\beta [N'/x]M'$.
6. If $M \equiv El(P)$, then $El(P) \Rightarrow_\beta El(P') \equiv M'$ and $P \Rightarrow_\beta P'$. By induction hypothesis, we have $[N/x]P \Rightarrow_\beta [N'/x]P'$. Hence $[N/x]M \Rightarrow_\beta [N'/x]M'$.
7. If $M \equiv (y : K_1)K_2$, then $(y : K_1)K_2 \Rightarrow_\beta (y : K_1')K_2' \equiv M'$ and $K_1 \Rightarrow_\beta K_1'$ and $K_2 \Rightarrow_\beta K_2'$. By induction hypothesis, we have $[N/x]K_1 \Rightarrow_\beta [N'/x]K_1'$ and $[N/x]K_2 \Rightarrow_\beta [N'/x]K_2'$. Hence $[N/x]M \Rightarrow_\beta [N'/x]M'$.

□

From Lemma 5 (1), (2) and (3), we know that $\twoheadrightarrow_\beta$ is the reflexive, transitive closure of $\Rightarrow_\beta$. Therefore, to prove the property of Church-Rosser w.r.t. β-reduction, it suffices to show the "diamond property" of $\Rightarrow_\beta$, *i.e.* if $M \Rightarrow_\beta N_1$ and $M \Rightarrow_\beta N_2$ then there is a N_3 such that $N_1 \Rightarrow_\beta N_3$ and $N_2 \Rightarrow_\beta N_3$. But we will prove a stronger statement in Lemma 6.

Definition 5. ***(Complete development for*** β***)*** *We define a map* $cd :: \Lambda \cup \Pi \to \Lambda \cup \Pi$.

1. $cd(x) = x$
2. $cd(MN) = \begin{cases} x\,cd(N) & \text{if } M \equiv x \\ cd(PQ)cd(N) & \text{if } M \equiv PQ \\ [cd(N)/x]cd(R) & \text{if } M \equiv \lambda x : K.R \end{cases}$
3. $cd(\lambda x : K.M) = \lambda x : cd(K).cd(M)$
4. $cd(Type) = Type$
5. $cd(El(M)) = El(cd(M))$
6. $cd((x : K_1)K_2) = (x : cd(K_1))cd(K_2)$

Having defined parallel reduction $\Rightarrow_\beta$ and complete development cd, we prove the following lemma.

Lemma 6. *If* $M \Rightarrow_\beta N$ *then* $N \Rightarrow_\beta cd(M)$.

Proof. By induction on M.

1. If $M \equiv x$, then $x \Rightarrow_\beta x \equiv N = cd(M)$.
2. If $M \equiv \lambda x : K.P$, then $\lambda x : K.P \Rightarrow_\beta \lambda x : K'.P' \equiv N$ and $K \Rightarrow_\beta K'$ and $P \Rightarrow_\beta P'$. By induction hypothesis, we have $K' \Rightarrow_\beta cd(K)$ and $P' \Rightarrow_\beta cd(P)$. Hence

$$N \Rightarrow_\beta \lambda x : cd(K).cd(P) = cd(M)$$

3. If $M \equiv xP$, then $xP \Rightarrow_\beta xQ \equiv N$ and $P \Rightarrow_\beta Q$. By induction hypothesis, we have $Q \Rightarrow_\beta cd(P)$. Hence

$$N \Rightarrow_\beta xcd(P) = cd(M)$$

4. If $M \equiv (PQ)R$, then $(PQ)R \Rightarrow_\beta M'R' \equiv N$ and $PQ \Rightarrow_\beta M'$ and $R \Rightarrow_\beta R'$. By induction hypothesis, we have $M' \Rightarrow_\beta cd(PQ)$ and $R' \Rightarrow_\beta cd(R)$. Hence

$$N \Rightarrow_\beta cd(PQ)cd(R) = cd(M)$$

5. If $M \equiv (\lambda x : K.P)Q$, then there are two sub-cases.
   (a) $(\lambda x : K.P)Q \Rightarrow_\beta (\lambda x : K'.P')Q' \equiv N$ and $K \Rightarrow_\beta K'$, $P \Rightarrow_\beta P'$ and $Q \Rightarrow_\beta Q'$. By induction hypothesis, we have $P' \Rightarrow_\beta cd(P)$ and $Q' \Rightarrow_\beta cd(Q)$. Hence

   $$N \equiv (\lambda x : K'.P')Q' \Rightarrow [cd(Q)/x]cd(P) = cd(M)$$

   (b) $(\lambda x : K.P)Q \Rightarrow_\beta [Q'/x]P' \equiv N$ and $P \Rightarrow_\beta P'$ and $Q \Rightarrow_\beta Q'$. By induction hypothesis, we have $P' \Rightarrow_\beta cd(P)$ and $Q' \Rightarrow_\beta cd(Q)$. By Lemma 5 (4), we have $[Q'/x]P' \Rightarrow_\beta [cd(Q)/x]cd(P)$. Hence $N \Rightarrow_\beta cd(M)$.
6. If $M \equiv Type$, then $Type \Rightarrow_\beta Type \equiv N = cd(M)$.
7. If $M \equiv El(P)$, then $El(P) \Rightarrow_\beta El(P') \equiv N$ and $P \Rightarrow_\beta P'$. By induction hypothesis, we have $P' \Rightarrow_\beta cd(P)$. Hence

$$N \Rightarrow_\beta El(cd(P)) = cd(M)$$

8. If $M \equiv (x : K_1)K_2$, then $(x : K_1)K_2 \Rightarrow_\beta (x : K_1')K_2' \equiv N$ and $K_1 \Rightarrow_\beta K_1'$ and $K_2 \Rightarrow_\beta K_2'$. By induction hypothesis, we have $K_1' \Rightarrow_\beta cd(K_1)$ and $K_2' \Rightarrow_\beta cd(K_2)$. Hence

$$N \Rightarrow_\beta (x : cd(K_1))cd(K_2) = cd(M)$$

□

Theorem 2. ***(Church-Rosser for*** β***)*** *For an arbitrary term or kind M, if $M \twoheadrightarrow_\beta N_1$ and $M \twoheadrightarrow_\beta N_2$ then there is a N_3 such that $N_1 \twoheadrightarrow_\beta N_3$ and $N_2 \twoheadrightarrow_\beta N_3$.*

Proof. By Lemma 6 and the fact that $\twoheadrightarrow_\beta$ is the transitive closure of $\Rightarrow_\beta$.

Lemma 7. *For an arbitrary term or kind M, if $M \longrightarrow_{\eta'} N_1$ and $M \longrightarrow_{\eta'} N_2$ then there is a N_3 such that $N_1 \longrightarrow_{\eta'}^{=} N_3$ and $N_2 \longrightarrow_{\eta'}^{=} N_3$.*

Proof. By induction on M and analysing different cases of one-step η'-reduction. One interesting case is the following.

$M \equiv \lambda x : K.(y\overline{Q})x \longrightarrow_{\eta'} y\overline{Q} \equiv N_1$ and $M \longrightarrow_{\eta'} \lambda x : K'.(y\overline{Q})x \equiv N_2$. Then, let $N_3 \equiv N_1$ and we have $N_1 \longrightarrow_{\eta'}^{=} N_3$ and $N_2 \longrightarrow_{\eta'}^{=} N_3$.

Corollary 1. *For an arbitrary term or kind M, if $M \longrightarrow_{\eta'}^{=} N_1$ and $M \longrightarrow_{\eta'}^{=} N_2$ then there is a N_3 such that $N_1 \longrightarrow_{\eta'}^{=} N_3$ and $N_2 \longrightarrow_{\eta'}^{=} N_3$.*

Theorem 3. ***(Church-Rosser for η')*** *For an arbitrary term or kind M, if $M \twoheadrightarrow_{\eta'} N_1$ and $M \twoheadrightarrow_{\eta'} N_2$ then there is a N_3 such that $N_1 \twoheadrightarrow_{\eta'} N_3$ and $N_2 \twoheadrightarrow_{\eta'} N_3$.*

Proof. By Corollary 1 and the fact that $\twoheadrightarrow_{\eta}$ is the transitive closure of $\longrightarrow_{\eta'}^{=}$.

Lemma 8. *For an arbitrary term or kind M, if $M \longrightarrow_{\beta} N_1$ and $M \longrightarrow_{\eta'} N_2$ then there is a N_3 such that $N_1 \longrightarrow_{\eta'} N_3$ and $N_2 \longrightarrow_{\beta}^{=} N_3$.*

Proof. By induction on M and analysing different cases of one-step η' and β-reduction. One interesting case is the following.

$M \equiv \lambda x : K.(y\overline{Q})x \longrightarrow_{\eta'} y\overline{Q} \equiv N_2$ and $M \longrightarrow_{\beta} \lambda x : K'.(y\overline{Q})x \equiv N_1$. Then, let $N_3 \equiv N_2$ and we have $N_1 \longrightarrow_{\eta'} N_3$ and $N_2 \longrightarrow_{\beta}^{=} N_3$.

Theorem 4. ***(Commutation for $\beta\eta'$)*** *For an arbitrary term or kind M, if $M \twoheadrightarrow_{\beta} N_1$ and $M \twoheadrightarrow_{\eta'} N_2$ then there is a N_3 such that $N_1 \twoheadrightarrow_{\eta'} N_3$ and $N_2 \twoheadrightarrow_{\beta} N_3$.*

Proof. By Lemma 8.

Theorem 5. ***(Church-Rosser for $\beta\eta'$)*** *For an arbitrary term or kind M, if $M \twoheadrightarrow_{\beta\eta'} N_1$ and $M \twoheadrightarrow_{\beta\eta'} N_2$ then there is a N_3 such that $N_1 \twoheadrightarrow_{\beta\eta'} N_3$ and $N_2 \twoheadrightarrow_{\beta\eta'} N_3$.*

Proof. By Theorem 2, 3 and 4.

5 Conclusion

For the λ-calculus defined in Definition 1, the property of Church-Rosser w.r.t. β and η-reduction is not an easy matter, unlike the systems with β-reduction only or the simple λ-calculus with β and η-reduction. In this paper, we introduce a new eta-reduction η' and its one-step reduction. The property of Church-Rosser holds for β and η'-reduction, without the condition of well-typedness.

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Appendix

Inference rules for a dependently typed logical framework

$$\frac{}{<>\ \ valid} \qquad \frac{\Gamma \vdash K\ \ kind \qquad x \notin FV(\Gamma)}{\Gamma, x : K\ \ valid}$$

$$\frac{\Gamma\ \ valid}{\Gamma \vdash Type\ \ kind} \qquad \frac{\Gamma \vdash A : Type}{\Gamma \vdash El(A)\ \ kind}$$

$$\frac{\Gamma \vdash K\ \ kind \qquad \Gamma, x : K \vdash K'\ \ kind}{\Gamma \vdash (x : K)K'\ \ kind}$$

$$\frac{\Gamma, x : K, \Gamma'\ \ valid}{\Gamma, x : K, \Gamma' \vdash x : K}$$

$$\frac{\Gamma, x : K \vdash k : K'}{\Gamma \vdash \lambda x : K.k : (x : K)K'}$$

$$\frac{\Gamma \vdash k : K \qquad \Gamma \vdash K'\ \ kind}{\Gamma \vdash k : K'}\ (K =_{\beta\eta'} K')$$

$$\frac{\Gamma \vdash f : (x : K)K' \qquad \Gamma \vdash k : K}{\Gamma \vdash f(k) : [k/x]K'}$$