# Computer Science at Kent

**Appendix for**
**RedAlert: Determinacy Inference for Prolog**

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**Abstract**

Proofs of the mathematical foundations and propositions and theorems stated and used in (Kriener and King, 2011).

# 1 Introduction

In (Kriener and King, 2011) we present a determinacy inference for Prolog including *cut*. This inference is developed in as a static analysis, comprising of a number of concrete semantics and abstractions thereof. Three theorems and a number of propositions formalise the connections that hold between these. We present here formal justification for the mathematical backdrop used in (Kriener and King, 2011), as well as proofs of the propositions and theorems stated there.

# 2 Appendix - Proofs

## 2.1 $Con_{seq}^{\downarrow}$ is a complete lattice

### 2.1.1 Relation on $Con_{seq}^{\downarrow}$ is a partial order

**The relation is reflexive:** $\vec{\Theta} \sqsubseteq \vec{\Theta}$
*Observe that* $: \forall \vec{\Theta} \in Con_{seq}^{\downarrow}(\vec{\Theta} \subseteq_{pw} \vec{\Theta} \wedge \vec{\Theta} \in Sub_{|\vec{\Theta}|})$
*hence* $\forall \vec{\Theta} \in Con_{seq}^{\downarrow}(\vec{\Theta} \sqsubseteq \vec{\Theta})$
*by selecting* $\Phi = \Theta$


**The relation is transitive:** $\vec{\Theta_1} \sqsubseteq \vec{\Theta_2} \wedge \vec{\Theta_2} \sqsubseteq \vec{\Theta_3} \rightarrow \vec{\Theta_1} \sqsubseteq \vec{\Theta_3}$
$\forall \vec{\Theta}_1, \vec{\Theta}_2, \vec{\Theta}_3 \in Con_{seq}^{\downarrow}((\vec{\Theta}_1 \sqsubseteq \vec{\Theta}_2 \wedge \vec{\Theta}_2 \sqsubseteq \vec{\Theta}_3) \rightarrow (\vec{\Theta}_1 \sqsubseteq \vec{\Theta}_3))$
*let* $|\vec{\Theta}_1| = l, \ |\vec{\Theta}_2| = m, \ |\vec{\Theta}_3| = n,$
$l \leq m \leq n$
$(\vec{\Theta}_1 \sqsubseteq \vec{\Theta}_2) \rightarrow \exists \vec{\Phi}_1 \in Sub_l(\vec{\Theta}_2).(\vec{\Theta}_1 \subseteq_{pw} \vec{\Phi}_1)$
$(\vec{\Theta}_2 \sqsubseteq \vec{\Theta}_3) \rightarrow \exists \vec{\Phi}_2 \in Sub_m(\vec{\Theta}_3).(\vec{\Theta}_2 \subseteq_{pw} \vec{\Phi}_2)$
*since* $\vec{\Theta}_2 \subseteq_{pw} \vec{\Phi}_2$ *and* $\exists \vec{\Phi}_1 \in Sub_l(\vec{\Theta}_2).(\vec{\Theta}_1 \subseteq_{pw} \vec{\Phi}_1) : \ \exists \vec{\Phi}_3 \in Sub_l(\vec{\Phi}_2).(\vec{\Theta}_1 \subseteq_{pw} \vec{\Phi}_3)$
$Sub_l(\vec{\Phi}_2) \subseteq Sub_l(\vec{\Theta}_3)$
*hence* $\exists \vec{\Phi}_3 \in Sub_l(\vec{\Theta}_3).(\vec{\Theta}_1 \subseteq_{pw} \vec{\Phi}_3)$
*therefore* $\vec{\Theta}_1 \sqsubseteq \vec{\Theta}_3$


**The relation is anti-symmetric:** $\forall \vec{\Theta}_1, \vec{\Theta}_2 \in Con_{seq}^{\downarrow}(\vec{\Theta}_1 \sqsubseteq \vec{\Theta}_2 \wedge \vec{\Theta}_2 \sqsubseteq \vec{\Theta}_1 \rightarrow \vec{\Theta}_1 = \vec{\Theta}_2)$
*let* $|\vec{\Theta}_1| = m, \ |\vec{\Theta}_2| = n$
$(\vec{\Theta}_1 \sqsubseteq \vec{\Theta}_2) \rightarrow \exists \vec{\Phi}_1 \in Sub_m(\vec{\Theta}_2)$ *such that* $\vec{\Theta}_1 \subseteq_{pw} \vec{\Phi}_1$
$(\vec{\Theta}_2 \sqsubseteq \vec{\Theta}_1) \rightarrow \exists \vec{\Phi}_2 \in Sub_n(\vec{\Theta}_1)$ *such that* $\vec{\Theta}_2 \subseteq_{pw} \vec{\Phi}_2$
$|\vec{\Phi}_1| = m$ *and* $|\vec{\Phi}_1| \leq n$ *hence* $m \leq n$
$|\vec{\Phi}_2| = n$ *and* $|\vec{\Phi}_2| \leq m$ *hence* $n \leq m$
*hence* $m = n$ *(by anti* $-$ *symmetry of* $\leq$*)*
*hence* $\vec{\Phi}_1 = \vec{\Theta}_2$ *and* $\vec{\Phi}_2 = \vec{\Theta}_1$
*hence* $\vec{\Theta}_1 \subseteq_{pw} \vec{\Theta}_2$ *and* $\vec{\Theta}_2 \subseteq_{pw} \vec{\Theta}_1$
*therefore* $:$
$\vec{\Theta}_1 = \vec{\Theta}_2$ *(by anti* $-$ *symmetry of* $\subseteq_{pw}$*)*

### 2.1.2 The meet of two sequences is unique and therefore well defined:

First note that by the definition of $\sqcap$, $\vec{\Theta} \sqcap \vec{\Psi} \sqsubseteq \vec{\Theta}$ and $\vec{\Theta} \sqcap \vec{\Psi} \sqsubseteq \vec{\Psi}$.
Then show: $\forall \vec{\Theta}, \vec{\Psi}, \vec{\Gamma} \in Con_{seq}^{\downarrow} : \vec{\Gamma} \sqsubseteq \vec{\Theta} \wedge \vec{\Gamma} \sqsubseteq \vec{\Psi} \rightarrow \vec{\Gamma} \sqsubseteq (\vec{\Theta} \sqcap \vec{\Psi})$

$|\vec{\Theta}| = n,\ |\vec{\Psi}| = m,\ |\vec{\Gamma}| = k$

$\vec{\Gamma} \sqsubseteq \vec{\Theta} \to \exists \vec{\Theta}_1\ \in Sub_k(\vec{\Theta}).(\vec{\Gamma} \subseteq_{pw} \vec{\Theta}_1)$

$\vec{\Gamma} \sqsubseteq \vec{\Psi} \to \exists \vec{\Psi}_1\ \in Sub_k(\vec{\Psi}).(\vec{\Gamma} \subseteq_{pw} \vec{\Psi}_1)$

$|\vec{\Theta}_1| = k,\ |\vec{\Psi}_1| = k$

assume (without loss of generality): $n \geq m$, then: $|\vec{\Theta} \sqcap \vec{\Psi}| = l,\ l \leq m$

since $\vec{\Gamma} \sqsubseteq \vec{\Theta}$ and $\vec{\Gamma} \sqsubseteq \vec{\Psi}$, $k \leq m$ *(and $k \leq n$)*

since $\vec{\Gamma} \subseteq_{pw} \vec{\Theta}_1$ *and* $\vec{\Gamma} \subseteq_{pw} \vec{\Psi}_1$, $\vec{\Gamma} \subseteq_{pw} (\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1)$

hence $\vec{\Gamma} \sqsubseteq (\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1)$

$(\vec{\Psi}_1 \in Sub_k(\vec{\Psi})) \to (\vec{\Psi}_1 \sqsubseteq \vec{\Psi})$

$(\vec{\Theta}_1 \in Sub_k(\vec{\Theta})) \to (\vec{\Theta}_1 \sqsubseteq \vec{\Theta})$

$(\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1) \in \{\vec{X} \cap_{pw} \vec{\Psi}_1 \mid \vec{X} \in Sub_k(\vec{\Theta})\}$ *(since $\vec{\Theta}_1 \in Sub_k(\vec{\Theta})$)*

$(\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1) \subseteq_{pw} \bigcup_{pw}\{\vec{X} \cap_{pw} \vec{\Psi}_1 \mid \vec{X} \in Sub_k(\vec{\Theta})\}$

*(note that since $\vec{\Gamma} \in Con_{seq}^{\downarrow}$, $\vec{\Gamma}$ does not contain $\{false\}$*

*and since $\vec{\Gamma} \subseteq_{pw} (\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1)$, $\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1$ does not contain $\{false\}$*

*hence $(\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1) = trim(\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1))$*

$(\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1) \sqsubseteq (\vec{\Theta} \sqcap \vec{\Psi}_1)$

$(\vec{\Theta} \sqcap \vec{\Psi}_1) \sqsubseteq (\vec{\Theta} \sqcap \vec{\Psi})$ *(since $\vec{\Psi}_1 \sqsubseteq \vec{\Psi}$ and $\sqcap$ is monotonic)*

*therefore $\vec{\Gamma} \sqsubseteq (\vec{\Theta} \sqcap \vec{\Psi})$*

## 2.2   Cut-normal form

We transform Prolog predicates that are defined by any number of clauses, none of which contains a disjunction, into this form by constructing $G_1, G_2, G_3$ and $G_4$ as follows:

| $G_1$: | If no clause precedes the clause containing the first *cut*, set $G_1$ to post(*false*). |
|---|---|
| | Else, if a single clause precedes the clause containing the first *cut*, set $G_1$ to the body of this clause. |
| | Otherwise, define an auxiliary predicate to wrap up all clauses preceding the clause containing the first *cut* and set $G_1$ to a call to that predicate. |
| $G_2$: | If there is no *cut* in the predicate, set $G_2$ to post(*false*). |
| | Else, if no atom precedes the first *cut*, set $G_2$ to post(*true*). |
| | Otherwise, set $G_2$ to the compound goal before the first *cut*. |

| | |
|---|---|
| $G_3$: | If there is no *cut* in the predicate, set $G_3$ to any goal, e.g. post(*true*). |
| | Else, if no goal follows the first *cut*, set $G_3$ to post(*true*). |
| | Else, if the compound goal following the first *cut* does not contain another *cut*, set $G_3$ to that goal. |
| | Otherwise, define an auxiliary predicate to wrap up the compound goal following the first *cut* and set $G_3$ to a call to that predicate. |
| $G_4$: | If no clause follows the clause containing the first *cut*, set $G_4$ to post(*false*). |
| | Else, if a single, *cut*-free clause follows the clause containing the first *cut*, set $G_4$ to the body of this clause. |
| | Otherwise, define an auxiliary predicate to wrap up all clauses following the clause containing the first *cut* and set $G_4$ to a call to that predicate. |

## 2.3 Theorem 1: $\bigcup(\mathcal{F}_G[\![G]\!]\mu_P\vec{\Theta}) \subseteq \bigcup(\vec{\Theta}) \cap S_G[\![G]\!]$

Notice first that the following things hold:
$\bigcup(\vec{\Psi}) \subseteq \bigcup(trim(\vec{\Psi}))$
$\Downarrow(\Theta \cup \Phi) = \Downarrow\Theta \cup \Downarrow\Phi$
$\Downarrow(\Theta \cap \Phi) = \Downarrow\Theta \cap \Downarrow\Phi$
$\overline{\exists}_{\vec{y}}(\Theta \cup \Phi) = \overline{\exists}_{\vec{y}}(\Theta) \cup \overline{\exists}_{\vec{y}}(\Phi)$
$\overline{\exists}_{\vec{y}}(\Theta \cap \Phi) \subseteq \overline{\exists}_{\vec{y}}(\Theta) \cap \overline{\exists}_{\vec{y}}(\Phi)$
$\rho_{\vec{x},\vec{y}}(\Theta \cup \Phi) = \rho_{\vec{x},\vec{y}}\Theta \cup \rho_{\vec{x},\vec{y}}\Phi$
$\rho_{\vec{x},\vec{y}}(\Theta \cap \Phi) \subseteq \rho_{\vec{x},\vec{y}}\Theta \cap \rho_{\vec{x},\vec{y}}\Phi$
$\overline{\exists}_{\vec{y}}(\Downarrow\overline{\exists}_{\vec{y}}(\Theta)) = \overline{\exists}_{\vec{y}}(\Theta)$

Proof by induction on length of $\vec{\Theta}$:
Base Case: $\vec{\Theta} = [\,]$
$\bigcup(\mathcal{F}_G[\![G]\!]\mu_P[\,]) = \bigcup([\,]) = \emptyset$
$\bigcup([\,]) \cap S_G[\![G]\!] = \emptyset \cap S_G[\![G]\!] = \emptyset$
$\emptyset \subseteq \emptyset$
therefore: $\bigcup(\mathcal{F}_G[\![G]\!]\mu_P[\,]) \subseteq \bigcup([\,]) \cap S_G[\![G]\!]$

Induction Step:
Assume: $\bigcup(\mathcal{F}_G[\![G]\!]\mu_P\vec{\Theta}) \subseteq \bigcup(\vec{\Theta}) \cap S_G[\![G]\!]$
Show: $\bigcup(\mathcal{F}_G[\![G]\!]\mu_P(\Theta : \vec{\Theta})) \subseteq \bigcup(\Theta : \vec{\Theta}) \cap S_G[\![G]\!]$

Induction on structure of G:
Two base cases: (1) $G = \mathsf{post}(\phi)$, (2) $G = p(\vec{x})$

(1) $G = \mathsf{post}(\phi)$
Assume: $\bigcup(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_P\vec{\Theta}) \subseteq \bigcup(\vec{\Theta}) \cap S_G[\![\mathsf{post}(\phi)]\!]$
Show: $\bigcup(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_P(\Theta : \vec{\Theta})) \subseteq \bigcup(\Theta : \vec{\Theta}) \cap S_G[\![\mathsf{post}(\phi)]\!]$
$\bigcup(\Theta : \vec{\Theta}) \cap S_G[\![\mathsf{post})(\phi)]\!]$
$= (\Theta \cap S_G[\![\mathsf{post}(\phi)]\!]) \cup (\bigcup(\vec{\Theta}) \cap S_G[\![\mathsf{post}(\phi)]\!])$
$= (\Theta \cap \Downarrow\{\phi\}) \cup (\bigcup(\vec{\Theta}) \cap S_G[\![\mathsf{post}(\phi)]\!])$

$\bigcup(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_P(\Theta : \vec{\Theta}))$
$= \bigcup(trim(\Downarrow\{\phi\} \cap \Theta : \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_P\vec{\Theta}))$
$\subseteq \bigcup(\Downarrow\{\phi\} \cap \Theta : \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_P\vec{\Theta})$
$= (\Downarrow\{\phi\} \cap \Theta) \cup \bigcup(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_P\vec{\Theta})$
$\subseteq (\Downarrow\{\phi\} \cap \Theta) \cup (\bigcup(\vec{\Theta}) \cap S_G[\![\mathsf{post}(\phi)]\!])$
therefore: $\bigcup(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_P(\Theta : \vec{\Theta})) \subseteq \bigcup(\Theta : \vec{\Theta}) \cap S_G[\![\mathsf{post}(\phi)]\!]$

(2) $G = p(\vec{x})$
Assume (without loss of generality): $p(\vec{y}) \leftarrow G_1; G_2, !, G_3; G_4 \in P$
Assume: $\bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_P\vec{\Theta}) \subseteq \bigcup(\vec{\Theta}) \cap S_G[\![p(\vec{x})]\!]$
Show: $\bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_P(\Theta : \vec{\Theta})) \subseteq \bigcup(\Theta : \vec{\Theta}) \cap S_G[\![p(\vec{x})]\!]$

$\quad \bigcup(\Theta : \vec{\Theta}) \cap S_G[\![p(\vec{x})]\!]$
$= (\Theta \cap S_G[\![p(\vec{x})]\!]) \cup (\bigcup(\vec{\Theta}) \cap S_G[\![p(\vec{x})]\!])$
$= (\Theta \cap \Downarrow\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_H[\![p(\vec{y})]\!])) \cup (\bigcup(\vec{\Theta}) \cap S_G[\![p(\vec{x})]\!])$
$= (\Theta \cap \Downarrow\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\bar{\exists}_{\vec{y}}(S_G[\![G_1]\!] \cup S_G[\![G_2, G_3]\!] \cup S_G[\![G_4]\!]))) \cup (\bigcup(\vec{\Theta}) \cap S_G[\![p(\vec{x})]\!])$
$= (\Theta \cap \Downarrow\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_1]\!] \cup S_G[\![G_2, G_3]\!] \cup S_G[\![G_4]\!])) \cup (\bigcup(\vec{\Theta}) \cap S_G[\![p(\vec{x})]\!])$
$= (\Theta \cap (\Downarrow\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_1]\!]) \cup \Downarrow\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_2, G_3]\!]) \cup \Downarrow\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_4]\!]))$
$\quad \cup (\bigcup(\vec{\Theta}) \cap S_G[\![p(\vec{x})]\!])$

$\quad \bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_P(\Theta : \vec{\Theta}))$
$= \bigcup(\Downarrow\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mu\ (p(\vec{y}))\ \Downarrow\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta])) \cap \Theta : \mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$
$= (\bigcup(\Downarrow\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mu\ (p(\vec{y}))\ \Downarrow\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta]))) \cap \Theta) \cup \bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$
$= (\bigcup(\Downarrow\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\bar{\exists}_y(\mathcal{F}_G[\![G_1]\!]\mu[\Theta'] : \vec{\Psi}))) \cap \Theta) \cup \bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$
$\quad where\ \vec{\Psi} = \begin{cases} \mathcal{F}_G[\![G_3]\!]\mu[\Phi] & if\ \mathcal{F}_G[\![G_2]\!]\mu[\Theta'] = \Phi : \vec{\Phi} \\ \mathcal{F}_G[\![G_4]\!]\mu[\Theta'] & if\ \mathcal{F}_G[\![G_2]\!]\mu[\Theta'] = [] \end{cases}$
$\quad and\ \Theta' = \Downarrow\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}(\Theta)$
To be on the safe side, consider the sequence resulting from appending *both* possibilities for $\vec{\Psi}$, the union of which is certainly a superset of the above:
$\subseteq (\bigcup(\Downarrow\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\bar{\exists}_y(\mathcal{F}_G[\![G_1]\!]\mu[\Theta'] : \mathcal{F}_G[\![G_3]\!]\mu[\Phi] : \mathcal{F}_G[\![G_4]\!]\mu[\Theta']))) \cap \Theta) \cup \bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$
$\quad where\ \Phi : \vec{\Phi} = \mathcal{F}_G[\![G_2]\!]\mu[\Theta']$
Again, changing this to include all, rather than only the first, possibilities for $\mathcal{F}_G[\![G_2]\!]\mu[\Theta']$ will result in a safe over-approximation, i.e. a superset of the above: $\subseteq (\bigcup(\Downarrow\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\bar{\exists}_y(\mathcal{F}_G[\![G_1]\!]\mu[\Theta'] :$
$\mathcal{F}_G[\![G_3]\!]\mu(\mathcal{F}_G[\![G_2]\!]\mu[\Theta']) : \mathcal{F}_G[\![G_4]\!]\mu[\Theta']))) \cap \Theta)$
$\quad \cup \bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$
$= (\bigcup(\Downarrow\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu[\Theta'] : \mathcal{F}_G[\![G_3]\!]\mu(\mathcal{F}_G[\![G_2]\!]\mu[\Theta']) : \mathcal{F}_G[\![G_4]\!]\mu[\Theta'])) \cap \Theta) \cup \bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$
$= (\bigcup(\Downarrow\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu[\Theta']) : \Downarrow\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_3]\!]\mu(\mathcal{F}_G[\![G_2]\!]\mu[\Theta'])) : \Downarrow\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_4]\!]\mu[\Theta']))$
$\quad \cap \Theta) \cup \bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$
$= ((\bigcup(\Downarrow\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu[\Theta'])) \cup \bigcup(\Downarrow\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_3]\!]\mu(\mathcal{F}_G[\![G_2]\!]\mu[\Theta']))) \cup \bigcup(\Downarrow\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_4]\!]\mu[\Theta'])))$
$\quad \cap \Theta) \cup \bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$
$= (\Downarrow\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\bigcup(\mathcal{F}_G[\![G_1]\!]\mu[\Theta'])) \cup \Downarrow\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\bigcup(\mathcal{F}_G[\![G_3]\!]\mu(\mathcal{F}_G[\![G_2]\!]\mu[\Theta']))) \cup \Downarrow\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\bigcup(\mathcal{F}_G[\![G_4]\!]\mu[\Theta']))$
$\quad \cap \Theta) \cup \bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$
since: $\bigcup(\mathcal{F}_G[\![G_1]\!]\mu[\Theta']) \subseteq S_G[\![G_1]\!] \cap \Theta'$
and: $\bigcup(\mathcal{F}_G[\![G_2]\!]\mu[\Theta']) \subseteq S_G[\![G_2]\!] \cap \Theta'$
hence: $\bigcup(\mathcal{F}_G[\![G_3]\!]\mu(\mathcal{F}_G[\![G_2]\!]\mu[\Theta'])) \subseteq S_G[\![G_3]\!] \cap (S_G[\![G_2]\!] \cap \Theta')$
hence: $\bigcup(\mathcal{F}_G[\![G_3]\!]\mu(\mathcal{F}_G[\![G_2]\!]\mu[\Theta'])) \subseteq (S_G[\![G_3]\!] \cap S_G[\![G_2]\!]) \cap \Theta'$

hence: $\bigcup(\mathcal{F}_G[\![G_3]\!]\mu(\mathcal{F}_G[\![G_2]\!]\mu[\Theta'])) \subseteq S_G[\![G_2, G_3]\!] \cap \Theta'$

and: $\bigcup(\mathcal{F}_G[\![G_4]\!]\mu[\Theta']) \subseteq S_G[\![G_4]\!] \cap \Theta'$

using these, therefore, the above superset of $\bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_P(\Theta : \vec{\Theta}))$ is a subset of:

$\subseteq ((\downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_1]\!] \cap \Theta') \cup \downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_2, G_3]\!] \cap \Theta') \cup \downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_4]\!] \cap \Theta')) \cap \Theta)$
$\quad\quad \cup \bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$

since: $\Theta' = \downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}(\Theta)$, the following holds: $\downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\Theta') = \downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}(\Theta))$
$\quad\quad = \downarrow\!\bar{\exists}_{\vec{x}}(\Theta) \supseteq \Theta$

intersecting this with $\Theta$ therefore gives $\Theta$ itself: $\downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\Theta') \cap \Theta = \Theta$ distributing the projections and collecting and intersection the occurrences of $\downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\Theta')$ and $\Theta$ above therefore gives:

$\subseteq ((\downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_1]\!]) \cup \downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_2, G_3]\!]) \cup \downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_4]\!])) \cap \Theta) \cup \bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$
$\subseteq ((\downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_1]\!]) \cup \downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_2, G_3]\!]) \cup \downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_4]\!])) \cap \Theta)$
$\quad\quad \cup (\bigcup(\vec{\Theta}) \cap S_G[\![p(\vec{x})]\!])$
$= \bigcup(\Theta : \vec{\Theta}) \cap S_G[\![p(\vec{x})]\!]$

Induction Step: $G = G_1, G_2$
Assume: $\bigcup(\mathcal{F}_G[\![G_1, G_2]\!]\mu_P\vec{\Theta}) \subseteq \bigcup(\vec{\Theta}) \cap S_G[\![G_1, G_2]\!]$
And: $\bigcup(\mathcal{F}_G[\![G_1]\!]\mu_P\vec{\Phi}) \subseteq \bigcup(\vec{\Phi}) \cap S_G[\![G_1]\!]$
And: $\bigcup(\mathcal{F}_G[\![G_2]\!]\mu_P\vec{\Phi}) \subseteq \bigcup(\vec{\Phi}) \cap S_G[\![G_2]\!]$
Show: $\bigcup(\mathcal{F}_G[\![G_1, G_2]\!]\mu_P(\Theta : \vec{\Theta})) \subseteq \bigcup(\Theta : \vec{\Theta}) \cap S_G[\![G_1, G_2]\!]$

$\quad\quad \bigcup(\Theta : \vec{\Theta}) \cap S_G[\![G_1, G_2]\!]$
$= (\Theta \cap S_G[\![G_1, G_2]\!]) \cup (\bigcup(\vec{\Theta}) \cap S_G[\![G_1, G_2]\!])$
$= (\Theta \cap S_G[\![G_1]\!] \cap S_G[\![G_2]\!]) \cup (\bigcup(\vec{\Theta}) \cap S_G[\![G_1, G_2]\!])$

$\quad\quad \bigcup(\mathcal{F}_G[\![G_1, G_2]\!]\mu_P(\Theta : \vec{\Theta}))$
$= \bigcup(\mathcal{F}_G[\![G_2]\!]\mu(\mathcal{F}_G[\![G_1]\!]\mu(\Theta : \vec{\Theta})))$
$\subseteq \bigcup(\mathcal{F}_G[\![G_1]\!]\mu(\Theta : \vec{\Theta}) \cap S_G[\![G_2]\!]$
$\subseteq \bigcup(\Theta : \vec{\Theta}) \cap S_G[\![G_1]\!] \cap S_G[\![G_2]\!]$
$= \bigcup(\Theta : \vec{\Theta}) \cap S_G[\![G_1, G_2]\!]$
QED

## 2.4 Theorem 2: For $\Theta \in Con^\downarrow$ and stratified $P = P_0 \cup \ldots \cup P_n$: $\Theta \subseteq \mathcal{D}_G[\![G]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G]\!]\mu_P[\Theta]| \leq 1$.

### 2.4.1 Lemma 1: $(\mathcal{F}_G[\![G]\!]\mu\vec{\Theta}) \cap \Psi = \mathcal{F}_G[\![G]\!]\mu(\vec{\Theta} \cap \Psi)$

Proof by nested induction on:
1. $\mu$,
2. $|\vec{\Theta}|$,
3. structure of $G$

1 Base Case: $\mu = \mu_\perp$
Show: $(\mathcal{F}_G[\![G]\!]\mu_\perp\vec{\Theta}) \cap \Psi = \mathcal{F}_G[\![G]\!]\mu_\perp(\vec{\Theta} \cap \Psi)$

1.1 Base Case: $\vec{\Theta} = []$
Show: $(\mathcal{F}_G[\![G]\!]\mu_\perp[]) \cap \Psi = \mathcal{F}_G[\![G]\!]\mu_\perp([] \cap \Psi)$
$([]) \cap \Psi = \mathcal{F}_G[\![G]\!]\mu_\perp([])$
$[] = []$

1.2 Induction Step: $(\Theta : \vec{\Theta})$
Assume: $(\mathcal{F}_G[\![H]\!]\mu_\perp \vec{\Theta}) \cap \Psi = \mathcal{F}_G[\![H]\!]\mu_\perp(\vec{\Theta} \cap \Psi)$
Show: $(\mathcal{F}_G[\![G]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![G]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi)$

1.2.1 Two Base Cases: (1) $G = \mathsf{post}(\phi)$, (2) $G = p(\vec{x})$

(1) $G = \mathsf{post}(\phi)$
Show: $(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi)$

$(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi)$
$trim(\downarrow\!\{\phi\} \cap \Theta : \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp\vec{\Theta}) \cap \Psi = \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp((\Theta \cap \Psi) : (\vec{\Theta} \cap \Psi))$
$(trim[\downarrow\!\{\phi\} \cap \Theta] : trim(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp\vec{\Theta})) \cap \Psi$
$\quad = trim(\downarrow\!\{\phi\} \cap \Theta \cap \Psi : \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp(\vec{\Theta} \cap \Psi))$
$(trim[\downarrow\!\{\phi\} \cap \Theta] \cap \Psi) : (trim(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp\vec{\Theta}) \cap \Psi)$
$\quad = trim[\downarrow\!\{\phi\} \cap \Theta \cap \Psi] : trim(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp(\vec{\Theta} \cap \Psi))$
by assumption: $(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp\vec{\Theta}) \cap \Psi = \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp(\vec{\Theta} \cap \Psi)$
$trim(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp\vec{\Theta}) \cap \Psi = trim(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp(\vec{\Theta} \cap \Psi))$
$trim(\downarrow\!\{\phi\} \cap \Theta) \cap \Psi = trim(\downarrow\!\{\phi\} \cap \Theta \cap \Psi)$

(2) $G = p(\vec{x})$
Show: $(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi)$
$(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi$
$= (\downarrow\!\rho_{\vec{y},\vec{x}}\exists_{\vec{y}}(\mu_\perp(p(\vec{y})) \downarrow\!\rho_{\vec{x},\vec{y}}\exists_{\vec{x}}([\Theta])) \cap \Theta : \mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp\vec{\Theta}) \cap \Psi$
$= (\downarrow\!\rho_{\vec{y},\vec{x}}\exists_{\vec{y}}([]) \cap \Theta : \mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp\vec{\Theta}) \cap \Psi$
$= ([]) : (\mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp\vec{\Theta}) \cap \Psi$
by assumption: $(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp\vec{\Theta}) \cap \Psi = \mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp(\vec{\Theta} \cap \Psi)$

$\mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi)$
$= \mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp((\Theta \cap \Psi) : (\vec{\Theta} \cap \Psi))$
$= \downarrow\!\rho_{\vec{y},\vec{x}}\exists_{\vec{y}}(\mu_\perp(p(\vec{y})) \downarrow\!\rho_{\vec{x},\vec{y}}\exists_{\vec{x}}([\Theta \cap \Psi])) \cap \Theta \cap \Psi : \mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp(\vec{\Theta} \cap \Psi)$
$= \downarrow\!\rho_{\vec{y},\vec{x}}\exists_{\vec{y}}([]) \cap \Theta \cap \Psi : \mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp(\vec{\Theta} \cap \Psi)$
$= ([]) : \mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp(\vec{\Theta} \cap \Psi)$
hence: $(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi)$

1.2.2 Induction Step: $G = G_1, G_2$
Assume: $(\mathcal{F}_G[\![G_1]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![G_1]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi)$
And: $(\mathcal{F}_G[\![G_2]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![G_2]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi)$
Show: $(\mathcal{F}_G[\![G_1, G_2]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![G_1, G_2]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi)$
$(\mathcal{F}_G[\![G_1, G_2]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi$
$= (\mathcal{F}_G[\![G_2]\!]\mu_\perp(\mathcal{F}_G[\![G_1]\!]\mu_\perp(\Theta : \vec{\Theta}))) \cap \Psi$
$= (\mathcal{F}_G[\![G_2]\!]\mu_\perp(\mathcal{F}_G[\![G_1]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi)$
$= \mathcal{F}_G[\![G_2]\!]\mu_\perp(\mathcal{F}_G[\![G_1]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi))$
$= \mathcal{F}_G[\![G_1, G_2]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi)$

2 Induction Step: $\mu = \mu_{k+1}$
Assume: $(\mathcal{F}_G[\![H]\!]\mu_k\vec{\Delta}) \cap \Lambda = \mathcal{F}_G[\![H]\!]\mu_k(\vec{\Delta} \cap \Lambda)$

Show: $(\mathcal{F}_G[\![G]\!]\mu_{k+1}\vec{\Theta}) \cap \Psi = \mathcal{F}_G[\![G]\!]\mu_{k+1}(\vec{\Theta} \cap \Psi)$
where $\mu_{k+1} = \mathcal{F}_P[\![P]\!]\mu_k$

2.1 Base Case: $\vec{\Theta} = []$
Show: $(\mathcal{F}_G[\![G]\!]\mu_{k+1}[]) \cap \Psi = \mathcal{F}_G[\![G]\!]\mu_{k+1}([] \cap \Psi)$
$([]) \cap \Psi = \mathcal{F}_G[\![G]\!]\mu_{k+1}([])$
$([]) = ([])$

2.2 Induction Step: $\vec{\Theta} = (\Theta : \vec{\Theta})$
Assume: $(\mathcal{F}_G[\![G]\!]\mu_{k+1}\vec{\Theta}) \cap \Psi = \mathcal{F}_G[\![G]\!]\mu_{k+1}(\vec{\Theta} \cap \Psi)$
Show: $(\mathcal{F}_G[\![G]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![G]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi)$

2.2.1 Two Base Cases: (1) $G = \mathsf{post}(\phi)$, (2) $G = p(\vec{x})$
(1) $G = \mathsf{post}(\phi)$
Show: $(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi)$

$(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi)$
$trim(\Downarrow\!\{\phi\} \cap \Theta : \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}\vec{\Theta}) \cap \Psi = \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}((\Theta \cap \Psi) : (\vec{\Theta} \cap \Psi))$
$(trim[\Downarrow\!\{\phi\} \cap \Theta] : trim(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}\vec{\Theta})) \cap \Psi$
$= trim(\Downarrow\!\{\phi\} \cap \Theta \cap \Psi : \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}(\vec{\Theta} \cap \Psi))$
$(trim(\Downarrow\!\{\phi\} \cap \Theta) \cap \Psi) : (trim(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}\vec{\Theta}) \cap \Psi)$
$= trim(\Downarrow\!\{\phi\} \cap \Theta \cap \Psi) : trim(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}(\vec{\Theta} \cap \Psi))$
by assumption: $\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}\vec{\Theta}) \cap \Psi = \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}(\vec{\Theta} \cap \Psi)$
hence: $trim(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}\vec{\Theta}) \cap \Psi = trim(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}(\vec{\Theta} \cap \Psi))$
$trim(\Downarrow\!\{\phi\} \cap \Theta) \cap \Psi = trim(\Downarrow\!\{\phi\} \cap \Theta \cap \Psi)$

(2) $G = p(\vec{x})$
Assume (without loss of generality): $p(\vec{y}) \leftarrow G_1; G_2, !, G_3; G_4 \in P$
Show: $(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi)$

$(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi$
$= (\Downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y}))\Downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta])) \cap \Theta : \mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}\vec{\Theta}) \cap \Psi$
$= (\Downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y}))\Downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta])) \cap \Theta) \cap \Psi : (\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}\vec{\Theta}) \cap \Psi$

$\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi)$
$= \mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}((\Theta \cap \Psi) : (\vec{\Theta} \cap \Psi))$
$= (\Downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y}))\Downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta \cap \Psi])) \cap \Theta \cap \Psi) : (\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}(\vec{\Theta} \cap \Psi))$
by assumption: $(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}\vec{\Theta}) \cap \Psi = (\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}(\vec{\Theta} \cap \Psi))$
hence the question is whether the following holds:
$(\Downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y}))\Downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta])) \cap \Theta) \cap \Psi$
$\quad = (\Downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y}))\Downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta \cap \Psi])) \cap \Theta \cap \Psi)$

$(\Downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y}))\Downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta \cap \Psi])) \cap \Theta) \cap \Psi$
$= \Downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu_k \Downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta \cap \Psi])) : \vec{\Delta})) \cap \Theta \cap \Psi$
$where \ \vec{\Delta} = \begin{cases} \mathcal{F}_G[\![G_3]\!]\mu_k[\Lambda] & if \ \mathcal{F}_G[\![G_2]\!]\mu_k \Downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta \cap \Psi]) = \Lambda : \vec{\Lambda} \\ \mathcal{F}_G[\![G_4]\!]\mu_k \Downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta \cap \Psi]) & if \ \mathcal{F}_G[\![G_2]\!]\mu_k \Downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta \cap \Psi]) = [] \end{cases}$
$= (\Downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu_k \Downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta \cap \Psi]))) \cap \Theta \cap \Psi) : (\Downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\vec{\Delta}) \cap \Theta \cap \Psi)$

Observe that for any $F$: $\downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![F]\!]\mu_k \downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta])) = \downarrow\!\bar{\exists}_{\vec{x}}(\mathcal{F}_G[\![F]\!]\mu_k \downarrow\!\bar{\exists}_{\vec{x}}([\Theta]))$

hence: $(\downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta \cap \Psi]))) \cap \Theta \cap \Psi$

$= (\downarrow\!\bar{\exists}_{\vec{x}}(\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\!\bar{\exists}_{\vec{x}}([\Theta \cap \Psi]))) \cap \Theta \cap \Psi$

$= (\downarrow\!\bar{\exists}_{\vec{x}}(\mathcal{F}_G[\![G_1]\!]\mu_k [\downarrow\!\bar{\exists}_{\vec{x}}(\Theta \cap \Psi) \cap \downarrow\!\bar{\exists}_{\vec{x}}(\Theta)])) \cap \Theta \cap \Psi$

$\qquad$ (since $\downarrow\!\bar{\exists}_{\vec{x}}(\Theta \cap \Psi) \subseteq \downarrow\!\bar{\exists}_{\vec{x}}(\Theta)$)

which by assumption is equal to:

$(\downarrow\!\bar{\exists}_{\vec{x}}(\mathcal{F}_G[\![G_1]\!]\mu_k [\downarrow\!\bar{\exists}_{\vec{x}}(\Theta)]) \cap \downarrow\!\bar{\exists}_{\vec{x}}(\Theta \cap \Psi)) \cap \Theta \cap \Psi$

$= (\downarrow\!\bar{\exists}_{\vec{x}}(\downarrow\!\bar{\exists}_{\vec{x}}(\mathcal{F}_G[\![G_1]\!]\mu_k [\downarrow\!\bar{\exists}_{\vec{x}}(\Theta)]) \cap \downarrow\!\bar{\exists}_{\vec{x}}(\Theta \cap \Psi)) \cap \Theta \cap \Psi$

$\qquad$ (since $\downarrow\!\bar{\exists}_{\vec{x}}(A \cap B) = \downarrow\!\bar{\exists}_{\vec{x}}(\downarrow\!\bar{\exists}_{\vec{x}}(A) \cap B)$)

$= (\downarrow\!\bar{\exists}_{\vec{x}}(\mathcal{F}_G[\![G_1]\!]\mu_k [\downarrow\!\bar{\exists}_{\vec{x}}(\Theta)]) \cap \downarrow\!\bar{\exists}_{\vec{x}}(\Theta \cap \Psi) \cap \Theta \cap \Psi$

$\qquad$ (since $\downarrow\!\bar{\exists}_{\vec{x}}(\downarrow\!\bar{\exists}_{\vec{x}}(A) \cap \downarrow\!\bar{\exists}_{\vec{x}}(B)) = \downarrow\!\bar{\exists}_{\vec{x}}(A) \cap \downarrow\!\bar{\exists}_{\vec{x}}(B)$)

$= (\downarrow\!\bar{\exists}_{\vec{x}}(\mathcal{F}_G[\![G_1]\!]\mu_k [\downarrow\!\bar{\exists}_{\vec{x}}(\Theta)]) \cap \Theta \cap \Psi$

$\qquad$ (since $\Theta \cap \Psi \subseteq \downarrow\!\bar{\exists}_{\vec{x}}(\Theta \cap \Psi)$)

$= (\downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta]))) \cap \Theta \cap \Psi$

by parallel reasoning:

$(\downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_4]\!]\mu_k \downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta \cap \Psi]))) \cap \Theta \cap \Psi$

$\qquad = (\downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_4]\!]\mu_k \downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta]))) \cap \Theta \cap \Psi$

also by parallel reasoning:

$(\downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_2]\!]\mu_k \downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta \cap \Psi]))) \cap \Theta \cap \Psi$

$\qquad = (\downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_2]\!]\mu_k \downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta]))) \cap \Theta \cap \Psi$

hence if $(\downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_2]\!]\mu_k \downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta \cap \Psi]))) \cap \Theta \cap \Psi = \Lambda : \vec{\Lambda}$

and $(\downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_2]\!]\mu_k \downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta]))) \cap \Theta \cap \Psi = \Phi : \vec{\Phi}$

then $\Lambda = \Phi$

hence $(\downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_3]\!]\mu_k [\Lambda]))) \cap \Theta \cap \Psi = (\downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_3]\!]\mu_k [\Phi]))) \cap \Theta \cap \Psi$

now say $\vec{\Gamma} = \begin{cases} \mathcal{F}_G[\![G_3]\!]\mu_k[\Phi] & \text{if } \mathcal{F}_G[\![G_2]\!]\mu_k \downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta]) = \Phi : \vec{\Phi} \\ \mathcal{F}_G[\![G_4]\!]\mu_k \downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta]) & \text{if } \mathcal{F}_G[\![G_2]\!]\mu_k \downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta]) = [] \end{cases}$

then $\vec{\Gamma} = \vec{\Delta}$

hence: $\downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta \cap \Psi])) : \vec{\Delta})) \cap \Theta \cap \Psi$

$= \downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta])) : \vec{\Gamma})) \cap \Theta \cap \Psi$

hence: $(\downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y})) \downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta])) \cap \Theta) \cap \Psi$

$= (\downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y})) \downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta \cap \Psi])) \cap \Theta \cap \Psi)$

therefore: $(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi)$

2.2.2 Induction Step $G = G_1, G_2$

Assume: $(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![G_1]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi)$

And: $(\mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![G_2]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi)$

Show: $(\mathcal{F}_G[\![G_1, G_2]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![G_1, G_2]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi)$

$(\mathcal{F}_G[\![G_1, G_2]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi$

$= (\mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}(\Theta : \vec{\Theta}))) \cap \Psi$

$= (\mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi)$

$= \mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi))$

$= \mathcal{F}_G[\![G_1, G_2]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi)$

QED

**2.4.2 Lemma 2:** $\mathcal{F}_G[\![G]\!]\mu[\Theta] = \mathcal{F}_G[\![G]\!]\mu[\Theta \cap S_G[\![G]\!]]$

Proof in two stages:
(a) $\mathcal{F}_G[\![G]\!]\mu[\Theta \cap S_G[\![G]\!]] \sqsubseteq \mathcal{F}_G[\![G]\!]\mu[\Theta]$
(b) $\mathcal{F}_G[\![G]\!]\mu[\Theta] \sqsubseteq \mathcal{F}_G[\![G]\!]\mu[\Theta \cap S_G[\![G]\!]]$

(a) by monotonicity of $\mathcal{F}_G$:
$[\Theta \cap S_G[\![G]\!]] \sqsubseteq [\Theta] \Rightarrow \mathcal{F}_G[\![G]\!]\mu[\Theta \cap S_G[\![G]\!]] \sqsubseteq \mathcal{F}_G[\![G]\!]\mu[\Theta]$

(b) $\mathcal{F}_G[\![G]\!]\mu[\Theta] \sqsubseteq \mathcal{F}_G[\![G]\!]\mu[\Theta \cap S_G[\![G]\!]]$
Proof by nested induction on:
1. $\mu$,
2. structure of $G$:

1 Base Case: $\mathcal{F}_G[\![G]\!]\mu_\perp[\Theta] \sqsubseteq \mathcal{F}_G[\![G]\!]\mu_\perp[\Theta \cap S_G[\![G]\!]]$

induction on structure of $G$:
1.1 Two Base Cases: (1) $G = \mathsf{post}(\phi)$, (2) $G = p(\vec{x})$
(1) $G = \mathsf{post}(\phi)$
Show: $\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp[\Theta] \sqsubseteq \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp[\Theta \cap S_G[\![\mathsf{post}(\phi)]\!]]$

$\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp[\Theta] = trim([\Theta \cap {\Downarrow}\{\phi\}])$
$\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp[\Theta \cap S_G[\![\mathsf{post}(\phi)]\!]]$
$\quad = trim([\Theta \cap S_G[\![\mathsf{post}(\phi)]\!] \cap {\Downarrow}\{\phi\}])$
$\quad = trim([\Theta \cap {\Downarrow}\{\phi\} \cap {\Downarrow}\{\phi\}])$
$\quad = trim([\Theta \cap {\Downarrow}\{\phi\}])$

(2) $G = p(\vec{x})$
Show: $\mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp[\Theta] \sqsubseteq \mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp[\Theta \cap S_G[\![p(\vec{x})]\!]]$

$\mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp[\Theta] = [\,]$
$\mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp[\Theta \cap S_G[\![p(\vec{x})]\!]] = [\,]$

1.2 Induction Step: $G = G_1, G_2$
Assume: $\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta_1] \sqsubseteq \mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta_1 \cap S_G[\![G_1]\!]]$
And: $\mathcal{F}_G[\![G_2]\!]\mu_\perp[\Theta_2] \sqsubseteq \mathcal{F}_G[\![G_2]\!]\mu_\perp[\Theta_2 \cap S_G[\![G_2]\!]]$
Show: $\mathcal{F}_G[\![G_1, G_2]\!]\mu_\perp[\Theta] \sqsubseteq \mathcal{F}_G[\![G_1, G_2]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]]$

$\mathcal{F}_G[\![G_1, G_2]\!]\mu_\perp[\Theta] = \mathcal{F}_G[\![G_2]\!]\mu_\perp(\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta])$
by assumption: $\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta] \sqsubseteq \mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1]\!]]$
hence: $\mathcal{F}_G[\![G_2]\!]\mu_\perp(\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta]) \sqsubseteq \mathcal{F}_G[\![G_2]\!]\mu_\perp(\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1]\!]])$
by assumption:
$\mathcal{F}_G[\![G_2]\!]\mu_\perp(\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1]\!]])$
$\quad \sqsubseteq \mathcal{F}_G[\![G_2]\!]\mu_\perp((\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1]\!]]) \cap S_G[\![G_2]\!])$
by Lemma 1: $(\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1]\!]]) \cap S_G[\![G_2]\!]$
$\quad = \mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1]\!] \cap S_G[\![G_2]\!]]$
hence: $\mathcal{F}_G[\![G_2]\!]\mu_\perp((\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1]\!]]) \cap S_G[\![G_2]\!])$
$\quad = \mathcal{F}_G[\![G_2]\!]\mu_\perp(\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1]\!] \cap S_G[\![G_2]\!]])$

$= \mathcal{F}_G[\![G_1, G_2]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]]$

therefore: $\mathcal{F}_G[\![G_1, G_2]\!]\mu_\perp[\Theta] \sqsubseteq \mathcal{F}_G[\![G_1, G_2]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]]$


2 Induction Step:

Assume: $\mathcal{F}_G[\![H]\!]\mu_k[\Theta'] \sqsubseteq \mathcal{F}_G[\![H]\!]\mu_k[\Theta' \cap S_G[\![H]\!]]$

Show: $\mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta] \sqsubseteq \mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta \cap S_G[\![G]\!]]$

where $\mu_{k+1} = \mathcal{F}_P[\![P]\!]\mu_k$


induction on structure of G:

2.1 Two Base Cases: (1) $G = \mathsf{post}(\phi)$, (2) $G = p(\vec{x})$

(1) $G = \mathsf{post}(\phi)$

Show: $\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}[\Theta] \sqsubseteq \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}[\Theta \cap S_G[\![\mathsf{post}(\phi)]\!]]$

where $\mu_{k+1} = \mathcal{F}_P[\![P]\!]\mu_k$


$\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}[\Theta] = trim([\Theta \cap \mathord{\downarrow}\{\phi\}])$

$\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}[\Theta \cap S_G[\![\mathsf{post}(\phi)]\!]]$

$\quad = trim([\Theta \cap S_G[\![\mathsf{post}(\phi)]\!] \cap \mathord{\downarrow}\{\phi\}])$

$\quad = trim([\Theta \cap \mathord{\downarrow}\{\phi\} \cap \mathord{\downarrow}\{\phi\}])$

$\quad = trim([\Theta \cap \mathord{\downarrow}\{\phi\}])$


(2) $G = p(\vec{x})$

Assume (without loss of generality): $p(\vec{y}) \leftarrow G_1; G_2, !, G_3; G_4 \in P$

Show: $\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}[\Theta] \sqsubseteq \mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}[\Theta \cap S_G[\![p(\vec{x})]\!]]$

where: $\mu_{k+1} = \mathcal{F}_P[\![P]\!]\mu_k$

$\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}[\Theta \cap S_G[\![p(\vec{x})]\!]]$

$\quad = \mathord{\downarrow}\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y})) \mathord{\downarrow}\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]])) \cap \Theta \cap S_G[\![p(\vec{x})]\!]$

$\mu_{k+1}(p(\vec{y})) \mathord{\downarrow}\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]]) = \mathord{\downarrow}\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu_k \mathord{\downarrow}\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]]) : \vec{\Psi})$

where

$\vec{\Psi} = \begin{cases} \mathcal{F}_G[\![G_3]\!]\mu_k[\Phi] & \text{if } \mathcal{F}_G[\![G_2]\!]\mu_k \mathord{\downarrow}\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]]) = \Phi : \vec{\Phi} \\ \mathcal{F}_G[\![G_4]\!]\mu_k \mathord{\downarrow}\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]]) & \text{if } \mathcal{F}_G[\![G_2]\!]\mu_k \mathord{\downarrow}\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]]) = [] \end{cases}$

now: $S_G[\![p(\vec{x})]\!] = \mathord{\downarrow}\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_H[\![p(\vec{y})]\!])$

$\quad = \mathord{\downarrow}\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mathord{\downarrow}\bar{\exists}_{\vec{y}}(S_G[\![G_1]\!] \cup S_G[\![G_2, G_3]\!] \cup S_G[\![G_4]\!]))$

$\quad = \mathord{\downarrow}\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_1]\!] \cup S_G[\![G_2, G_3]\!] \cup S_G[\![G_4]\!])$

$\quad = \mathord{\downarrow}\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_1]\!]) \cup \mathord{\downarrow}\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_2, G_3]\!]) \cup \mathord{\downarrow}\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_4]\!])$

$\qquad$ (because $\bar{\exists}$ distributes over $\cup$)

Since $S_G[\![p(\vec{x})]\!]$ is the union of these three components, it is a superset of each of them, hence:

$S_G[\![p(\vec{x})]\!] \supseteq \mathord{\downarrow}\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_1]\!])$

and: $S_G[\![p(\vec{x})]\!] \supseteq \mathord{\downarrow}\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_2, G_3]\!])$

and: $S_G[\![p(\vec{x})]\!] \supseteq \mathord{\downarrow}\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_4]\!])$

Intersecting each side with $\Theta$ preserves the order, hence: $\Theta \cap S_G[\![p(\vec{x})]\!] \supseteq \Theta \cap \mathord{\downarrow}\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_1]\!])$

and: $\Theta \cap S_G[\![p(\vec{x})]\!] \supseteq \Theta \cap \mathord{\downarrow}\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_2, G_3]\!])$

and: $\Theta \cap S_G[\![p(\vec{x})]\!] \supseteq \Theta \cap \mathord{\downarrow}\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_4]\!])$

Again, projecting and renaming both sides in the same way preserves the order, hence: $\mathord{\downarrow}\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}(\Theta \cap S_G[\![p(\vec{x})]\!]) \supseteq \mathord{\downarrow}\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}(\Theta \cap \mathord{\downarrow}\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_1]\!]))$

and: $\mathord{\downarrow}\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}(\Theta \cap S_G[\![p(\vec{x})]\!]) \supseteq \mathord{\downarrow}\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}(\Theta \cap \mathord{\downarrow}\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_2, G_3]\!]))$

and: $\mathord{\downarrow}\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}(\Theta \cap S_G[\![p(\vec{x})]\!]) \supseteq \mathord{\downarrow}\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}(\Theta \cap \mathord{\downarrow}\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(S_G[\![G_4]\!]))$

Now, since the following holds in general:

$$\downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}(\Gamma_1 \cap \downarrow\rho_{\vec{y},\vec{x}}\vec{\exists}_{\vec{y}}(\Gamma_2)) \supseteq \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}(\Gamma_1) \cap \Gamma_2,$$
performing the same transformation on the above still preserves the order,

hence: $\downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}(\Theta \cap S_G[\![p(\vec{x})]\!]) \supseteq \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_1]\!]$

and: $\downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}(\Theta \cap S_G[\![p(\vec{x})]\!]) \supseteq \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_2, G_3]\!]$

$\qquad\qquad = \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_2]\!] \cap S_G[\![G_3]\!]$

and: $\downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}(\Theta \cap S_G[\![p(\vec{x})]\!]) \supseteq \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_4]\!]$

by monotonicity of $\mathcal{F}_G$, therefore:

$\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]]) \sqsupseteq \mathcal{F}_G[\![G_1]\!]\mu_k[\downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_1]\!]]$

by assumption: $\mathcal{F}_G[\![G_1]\!]\mu_k[\downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_1]\!]] \sqsupseteq \mathcal{F}_G[\![G_1]\!]\mu_k[\downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}(\Theta)]$

hence the following holds of the first part of the sequence:

$\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]]) \sqsupseteq \mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}[\Theta]$

and similarly: $\mathcal{F}_G[\![G_4]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]]) \sqsupseteq \mathcal{F}_G[\![G_4]\!]\mu_k[\downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_4]\!]]$

by assumption: $\mathcal{F}_G[\![G_4]\!]\mu_k[\downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_4]\!]] \sqsupseteq \mathcal{F}_G[\![G_4]\!]\mu_k[\downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}(\Theta)]$

hence the parallel thing holds for the second possibility of the second part of the sequence:

$\mathcal{F}_G[\![G_4]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]]) \sqsupseteq \mathcal{F}_G[\![G_4]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}[\Theta]$

As for the first possibility for the second part of the sequence, consider this:

by monotonicity of $\mathcal{F}_G$:

$\Phi : \vec{\Phi} = \mathcal{F}_G[\![G_2]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]])$

$\qquad\qquad \sqsupseteq \mathcal{F}_G[\![G_2]\!]\mu_k[\downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_2]\!] \cap S_G[\![G3]\!]]$

by assumption:

$\mathcal{F}_G[\![G_2]\!]\mu_k[\downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_2]\!] \cap S_G[\![G3]\!]] \sqsupseteq \mathcal{F}_G[\![G_2]\!]\mu_k[\downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G3]\!]]$

by Lemma 1: $\mathcal{F}_G[\![G_2]\!]\mu_k[\downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G3]\!]] = \mathcal{F}_G[\![G_2]\!]\mu_k[\downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}[\Theta] \cap S_G[\![G3]\!]$

hence: $\Phi : \vec{\Phi} \sqsupseteq \mathcal{F}_G[\![G_2]\!]\mu_k[\downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}[\Theta] \cap S_G[\![G3]\!]$

now call the part of the sequence we are aiming for here $\Lambda : \vec{\Lambda} = \mathcal{F}_G[\![G_2]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}([\Theta])$

then: $\Phi : \vec{\Phi} \sqsupseteq (\Lambda : \vec{\Lambda}) \cap S_G[\![G_3]\!]$

hence: $[\Phi] \sqsupseteq [\Lambda \cap S_G[\![G_3]\!]]$

hence: $\mathcal{F}_G[\![G_3]\!]\mu_k[\Phi] \sqsupseteq \mathcal{F}_G[\![G_3]\!]\mu_k[\Lambda \cap [\![G_3]\!]]$

by assumption: $\mathcal{F}_G[\![G_3]\!]\mu_k[\Lambda \cap [\![G_3]\!]] \sqsupseteq \mathcal{F}_G[\![G_3]\!]\mu_k[\Lambda]$

hence: $\mathcal{F}_G[\![G_3]\!]\mu_k[\Phi] \sqsupseteq \mathcal{F}_G[\![G_3]\!]\mu_k[\Lambda]$

These last few lines show that each part of the sequence we are considering is greater than the sequence we are aiming for. Pulling these together, we arrive at:

$\vec{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]]) : \vec{\Psi}) \sqsupseteq \vec{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}([\Theta]) : \vec{\Delta})$

where

$\vec{\Psi} = \begin{cases} \mathcal{F}_G[\![G_3]\!]\mu_k[\Phi] & \text{if } \mathcal{F}_G[\![G_2]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]]) = \Phi : \vec{\Phi} \\ \mathcal{F}_G[\![G_4]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]]) & \text{if } \mathcal{F}_G[\![G_2]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]]) = [] \end{cases}$

and $\vec{\Delta} = \begin{cases} \mathcal{F}_G[\![G_3]\!]\mu_k[\Lambda] & \text{if } \mathcal{F}_G[\![G_2]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}([\Theta]) = \Lambda : \vec{\Lambda} \\ \mathcal{F}_G[\![G_4]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}([\Theta]) & \text{if } \mathcal{F}_G[\![G_2]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}([\Theta]) = [] \end{cases}$

therefore: $\mu_{k+1}(p(\vec{y})) \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]]) \sqsupseteq \mu_{k+1}(p(\vec{y})) \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}([\Theta])$

applying the same renaming and projection to both sides preserves the order:

$\downarrow\rho_{\vec{y},\vec{x}}\vec{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y})) \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]])) \sqsupseteq \downarrow\rho_{\vec{y},\vec{x}}\vec{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y})) \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}([\Theta]))$

now name these two sequences:

$\downarrow\rho_{\vec{y},\vec{x}}\vec{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y})) \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]])) = \vec{\Psi'}$

and: $\downarrow\rho_{\vec{y},\vec{x}}\vec{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y})) \downarrow\rho_{\vec{x},\vec{y}}\vec{\exists}_{\vec{x}}([\Theta])) = \vec{\Delta'}$

and notice the following two facts:

(1) $\vec{\Delta'} \cap \Theta = \mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}[\Theta]$

(2) $\vec{\Psi'} \cap S_G[\![p(\vec{x})]\!] \cap \Theta = \mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}[\Theta \cap S_G[\![p(\vec{x})]\!]]$

then from above we have: $\vec{\Delta}' \sqsubseteq \vec{\Psi}'$

hence: $\vec{\Delta}' \cap \Theta \sqsubseteq \vec{\Psi}'$

by (1) and Theorem 1, therefore: $\bigcup(\vec{\Delta}') \cap \Theta = \bigcup(\vec{\Delta}' \cap \Theta) \subseteq \Theta \cap S_G[\![p(\vec{x})]\!]$

hence: $\bigcup(\vec{\Delta}') \subseteq S_G[\![p(\vec{x})]\!]$

therefore for each $\Delta'$ in $\vec{\Delta}'$: $\Delta' \subseteq S_G[\![p(\vec{x})]\!]$

hence for each $\Delta'$ in $\vec{\Delta}'$: $\Delta' \cap S_G[\![p(\vec{x})]\!] = \Delta'$

hence: $\vec{\Delta}' \cap S_G[\![p(\vec{x})]\!] = \vec{\Delta}'$

hence: $\vec{\Delta}' \cap \Theta = \vec{\Delta}' \cap S_G[\![p(\vec{x})]\!] \cap \Theta \sqsubseteq \vec{\Psi}' \cap S_G[\![p(\vec{x})]\!] \cap \Theta$

substituting using (2), we therefore arrive at:

$\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}[\Theta] \sqsubseteq \mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}[\Theta \cap S_G[\![p(\vec{x})]\!]]$

2.2 Induction Step: $G = G_1, G_2$

Assume: $\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta_1] \sqsubseteq \mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta_1 \cap S_G[\![G_1]\!]]$

And: $\mathcal{F}_G[\![G_2]\!]\mu_{k+1}[\Theta_2] \sqsubseteq \mathcal{F}_G[\![G_2]\!]\mu_{k+1}[\Theta_2 \cap S_G[\![G_2]\!]]$

Show: $\mathcal{F}_G[\![G_1, G_2]\!]\mu_{k+1}[\Theta] \sqsubseteq \mathcal{F}_G[\![G_1, G_2]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]$

$\mathcal{F}_G[\![G_1, G_2]\!]\mu_{k+1}[\Theta] = \mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta])$

by assumption: $\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta] \sqsubseteq \mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1]\!]]$

hence: $\mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta]) \sqsubseteq \mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1]\!]])$

by assumption:

$\mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1]\!]]) \sqsubseteq \mathcal{F}_G[\![G_2]\!]\mu_{k+1}((\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1]\!]]) \cap S_G[\![G_2]\!])$

by Lemma 1: $(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1]\!]]) \cap S_G[\![G_2]\!] = \mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1]\!] \cap S_G[\![G_2]\!]]$

hence: $\mathcal{F}_G[\![G_2]\!]\mu_{k+1}((\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1]\!]]) \cap S_G[\![G_2]\!])$

$\quad = \mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1]\!] \cap S_G[\![G_2]\!]])$

$\quad = \mathcal{F}_G[\![G_1, G_2]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]$

therefore: $\mathcal{F}_G[\![G_1, G_2]\!]\mu_{k+1}[\Theta] \sqsubseteq \mathcal{F}_G[\![G_1, G_2]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]$

Therefore since: (a) $\mathcal{F}_G[\![G]\!]\mu[\Theta \cap S_G[\![G]\!]] \sqsubseteq \mathcal{F}_G[\![G]\!]\mu[\Theta]$

and (b) $\mathcal{F}_G[\![G]\!]\mu[\Theta] \sqsubseteq \mathcal{F}_G[\![G]\!]\mu[\Theta \cap S_G[\![G]\!]]$,

it follows that: $\quad \mathcal{F}_G[\![G]\!]\mu[\Theta] = \mathcal{F}_G[\![G]\!]\mu[\Theta \cap S_G[\![G]\!]]$

QED

**2.4.3 Proof of Theorem 2: For $\Theta \in Con^{\downarrow}$ and stratified $P = P_0 \cup \ldots \cup P_n$: $\Theta \subseteq \mathcal{D}_G[\![G]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G]\!]\mu_P[\Theta]| \leq 1$.**

First notice that the following things hold:

(1) $\Theta \subseteq (\Phi \rightarrow \Psi) \Rightarrow \Theta \cap \Phi \subseteq \Psi$

(2) $\Theta \subseteq mux(\Phi, \Psi) \Rightarrow (\Theta \cap \Phi = \{false\}) \vee (\Theta \cap \Psi = \{false\})$

(3) $\mathcal{F}_G[\![G]\!]\mu\vec{\Theta} \subseteq \bigcup \vec{\Theta} \cap S_G[\![G]\!]$

for any $\mu$ constructed by application of $\mathcal{F}_P[\![P]\!]$ to $\mu_\perp$

(4) $\overline{\forall}_{\vec{y}}(\Theta \cap \Phi) = \overline{\forall}_{\vec{y}}(\Theta) \cap \overline{\forall}_{\vec{y}}(\Phi)$

(5) $\Theta \subseteq {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\Phi) \Rightarrow {\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \subseteq \Phi$

This holds due to the following few lines of reasoning:

$\Theta \subseteq \overline{\exists}_{\vec{x}}(\Theta)$ (since $\overline{\exists}$ is extensive)

if $\Theta \subseteq {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\Phi)$

then $\overline{\exists}_{\vec{x}}(\Theta) \subseteq \overline{\exists}_{\vec{x}}({\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\Phi))$

(by monotonicity of $\overline{\exists}$)

then ${\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \subseteq {\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}({\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\Phi)) = {\downarrow}\rho_{\vec{x},\vec{y}}({\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\Phi))$

(by monotonicity of $\downarrow, \rho$)

$\downarrow\rho_{\vec{x},\vec{y}}\downarrow\rho_{\vec{y},\vec{x}}$ cancel out and $\overline{\forall}$ is reductive, hence:

$\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \subseteq \Phi$

(6) $\mathcal{F}_G[\![G]\!]\mu[\Theta] \sqsubseteq \mathcal{F}_G[\![G]\!]\mu(\Theta : \vec{\Theta})$

again for any $\mu$ constructed by application of $\mathcal{F}_P[\![P]\!]$ to $\mu_\perp$

(7) $\vec{\Theta}_1 \sqsubseteq \vec{\Theta}_2 \Rightarrow |\vec{\Theta}_1| \leq |\vec{\Theta}_2|$


Proof by nested induction on:

1. $\mu$,

2. structure of $G$:

1 Base Case: $\mu = \mu_\perp$

show: $\Theta \subseteq \mathcal{D}_G[\![G]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G]\!]\mu_\perp[\Theta]| \leq 1$


Induction on structure of $G$:

1.1 Two Base Cases: (1) $G = \mathsf{post}(\phi)$, (2) $G = p(\vec{x})$


(1) $G = \mathsf{post}(\phi)$:

Show: $\Theta \subseteq \mathcal{D}_G[\![\mathsf{post}(\phi)]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp[\Theta]| \leq 1$


$\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp[\Theta] = trim([\downarrow\{\phi\} \cap \Theta])$

hence: $|\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp[\Theta]| = |trim([\downarrow\{\phi\} \cap \Theta])| \leq 1$


(2) $G = p(\vec{x})$

Show: $\Theta \subseteq \mathcal{D}_G[\![p(\vec{y})]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![p(\vec{y})]\!]\mu_\perp[\Theta]| \leq 1$


$\mathcal{F}_G[\![p(\vec{y})]\!]\mu_\perp[\Theta] = \downarrow\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mu_\perp\ (p(\vec{y}))\ \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])) \cap \Theta : []$

$\mu_\perp\ (p(\vec{y}))\ \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = []$

hence: $\mathcal{F}_G[\![p(\vec{y})]\!]\mu_\perp[\Theta] = []$

hence: $|\mathcal{F}_G[\![p(\vec{y})]\!]\mu_\perp[\Theta]| = |[]| = 0$


1.2 Induction Step:

$G = G_1, G_2$:

Assume: $\Theta_1 \subseteq \mathcal{D}_G[\![G_1]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta_1]| \leq 1$

And: $\Theta_2 \subseteq \mathcal{D}_G[\![G_2]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G_2]\!]\mu_\perp[\Theta_2]| \leq 1$

Show: $\Theta \subseteq \mathcal{D}_G[\![G]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G]\!]\mu_\perp[\Theta]| \leq 1$


$\mathcal{D}_G[\![G]\!]\delta_P = (S_G[\![G_2]\!] \to \mathcal{D}_G[\![G_1]\!]\delta_P) \cap (S_G[\![G_1]\!] \to \mathcal{D}_G[\![G_2]\!]\delta_P)$

$\Theta \subseteq \mathcal{D}_G[\![G]\!]\delta_P \Rightarrow \Theta \subseteq (S_G[\![G_1]\!] \to \mathcal{D}_G[\![G_2]\!]\delta_P) \Rightarrow \Theta \cap S_G[\![G_1]\!] \subseteq \mathcal{D}_G[\![G_2]\!]\delta_P$

$\Theta \subseteq \mathcal{D}_G[\![G]\!]\delta_P \Rightarrow \Theta \subseteq (S_G[\![G_2]\!] \to \mathcal{D}_G[\![G_1]\!]\delta_P) \Rightarrow \Theta \cap S_G[\![G_2]\!] \subseteq \mathcal{D}_G[\![G_1]\!]\delta_P$

$\mathcal{F}_G[\![G]\!]\mu_\perp[\Theta] = \mathcal{F}_G[\![G]\!]\mu_\perp[\Theta \cap S_G[\![G]\!]]$ *(by Lemma 2)*

$\mathcal{F}_G[\![G]\!]\mu_\perp[\Theta \cap S_G[\![G]\!]] = \mathcal{F}_G[\![G_2]\!]\mu_\perp(\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]])$

$\Theta \cap S_G[\![G_1, G_2]\!] = \Theta \cap S_G[\![G_1]\!] \cap S_G[\![G_2]\!] \subseteq \Theta \cap S_G[\![G_2]\!] \subseteq \mathcal{D}_G[\![G_1]\!]\delta_P$

hence by assumption: $|\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]]| \leq 1$

distinguish two cases:

(a) $|\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]]| = 0$,

(b) $|\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]]| = 1$

(a) $|\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]]| = 0$
$\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]] = []$
$\mathcal{F}_G[\![G]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]] = \mathcal{F}_G[\![G_2]\!]\mu_\perp[] = []$
hence: $|\mathcal{F}_G[\![G]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]]| \le 1$
by Lemma 2 (remembering $G = G_1, G_2$): $|\mathcal{F}_G[\![G]\!]\mu_\perp[\Theta]| \le 1$

(b) $|\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]]| = 1$
$\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]] = [\Psi]$
by Theorem 1: $\bigcup(\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]])$
$\qquad \subseteq \Theta \cap S_G[\![G_1, G_2]\!] \cap S_G[\![G_1]\!]$
$\qquad \subseteq \Theta \cap S_G[\![G_1]\!]$
hence: $\Psi \subseteq \Theta \cap S_G[\![G_1]\!]$
hence: $\Psi \subseteq \mathcal{D}_G[\![G_2]\!]\delta_P$
hence by assumption: $|\mathcal{F}_G[\![G_2]\!]\mu_\perp[\Psi]| \le 1$
hence (again by Lemma 2): $|\mathcal{F}_G[\![G_2]\!]\mu_\perp(\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]])|$
$\qquad = |\mathcal{F}_G[\![G]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]]|$
$\qquad = |\mathcal{F}_G[\![G]\!]\mu_\perp[\Theta]| \le 1$

2 Induction Step:
Assume: $X \subseteq \mathcal{D}_G[\![H]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![H]\!]\mu_k[X]| \le 1$
Show: $\Theta \subseteq \mathcal{D}_G[\![G]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta]| \le 1$
where $\mu_{k+1} = \mathcal{F}_P[\![P]\!]\mu_k$

Induction on structure of $G$:
2.1 Two base cases: (1) $G = \mathsf{post}(\phi)$, (2) $G = p(\vec{x})$

(1) $G = \mathsf{post}(\phi)$:
Show: $\Theta \subseteq \mathcal{D}_G[\![\mathsf{post}(\phi)]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}[\Theta]| \le 1$

$\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}[\Theta] = trim([\downarrow\!\{\phi\} \cap \Theta])$
hence: $|\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}[\Theta]| = |trim([\downarrow\!\{\phi\} \cap \Theta])| \le 1$

(2) $G = p(\vec{x})$
Assume (without loss of generality): $p(\vec{y}) \leftarrow G_1; G_2, !, G_3; G_4 \in P$
Show: $\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}[\Theta]| \le 1$

$\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}[\Theta] = \downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mu_{k+1}\ (p(\vec{y}))\ \downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta])) \cap \Theta$
hence: $|\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}[\Theta]| = |\downarrow\!\rho_{\vec{y},\vec{x}}\bar{\exists}_{\vec{y}}(\mu_{k+1}\ (p(\vec{y}))\ \downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta])) \cap \Theta| = |\mu_{k+1}\ (p(\vec{y}))\ \downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta])|$
and: $\mu_{k+1}\ (p(\vec{y}))\ \downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta]) = \bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu_k\downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta]) : \vec{\Psi})$
where $\vec{\Psi} = \begin{cases} \mathcal{F}_G[\![G_3]\!]\mu_k[\Phi] & \text{if } \mathcal{F}_G[\![G_2]\!]\mu_k\downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta]) = \Phi : \vec{\Phi} \\ \mathcal{F}_G[\![G_4]\!]\mu_k\downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta]) & \text{if } \mathcal{F}_G[\![G_2]\!]\mu_k\downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta]) = [] \end{cases}$

$|\bar{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu_k\downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta]) : \vec{\Psi})| = |\mathcal{F}_G[\![G_1]\!]\mu_k\downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta]) : \vec{\Psi}|$

Show $\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G_1]\!]\mu_k\downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta]) : \vec{\Psi}| \le 1$ in two steps:

1 Show that each component cannot be longer than 1:
1a Show: $\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G_1]\!]\mu_k\downarrow\!\rho_{\vec{x},\vec{y}}\bar{\exists}_{\vec{x}}([\Theta])| \le 1$

1b Show: $\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G_4]\!]\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \leq 1$

1c Show: $\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G_3]\!]\mu_k[\Phi]| \leq 1$

where $\mathcal{F}_G[\![G_2]\!]\mu_k {\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = \Phi : \vec{\Phi}$


2 Show that only one component can be longer than 0:

$\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow \neg(|\mathcal{F}_G[\![G_1]\!]\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \neq 0 \wedge |\vec{\Psi}| \neq 0)$

This is done thus:

2a Show:

$\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow \neg(|\mathcal{F}_G[\![G_1]\!]\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \neq 0 \wedge |\mathcal{F}_G[\![G_4]\!]\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \neq 0)$

2b Show:

$\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow \neg(|\mathcal{F}_G[\![G_1]\!]\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \neq 0 \wedge |\mathcal{F}_G[\![G_3]\!]\mu_k[\Phi]| \neq 0)$

where $\mathcal{F}_G[\![G_2]\!]\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = \Phi : \vec{\Phi}$


$\mathcal{D}_G[\![p(\vec{x})]\!]\delta_P = {\downarrow}\rho_{\vec{y},\vec{x}}(\overline{\forall}_{\vec{y}}(\delta_P(p(\vec{y}))))$

$= {\downarrow}\rho_{\vec{y},\vec{x}}(\overline{\forall}_{\vec{y}}(\overline{\downarrow}\overline{\forall}_{\vec{y}}(\mathcal{D}_G[\![G_1]\!]\delta_P \cap (S_G[\![G_2]\!] \rightarrow \mathcal{D}_G[\![G_3]\!]\delta_P) \cap \mathcal{D}_G[\![G_4]\!]\delta_P \cap \Theta_1 \cap \Theta_2)))$

$= {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\mathcal{D}_G[\![G_1]\!]\delta_P \cap (S_G[\![G_2]\!] \rightarrow \mathcal{D}_G[\![G_3]\!]\delta_P) \cap \mathcal{D}_G[\![G_4]\!]\delta_P \cap \Theta_1 \cap \Theta_2)$

   where $\Theta_1 = mux(S_G[\![G_1]\!], S_G[\![G_4]\!])$

   and $\Theta_2 = mux(S_G[\![G_1]\!], S_G[\![G_2, G_3]\!])$

$= {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\mathcal{D}_G[\![G_1]\!]\delta_P)$

   $\cap {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(S_G[\![G_2]\!] \rightarrow \mathcal{D}_G[\![G_3]\!]\delta_P)$

   $\cap {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\mathcal{D}_G[\![G_4]\!]\delta_P)$

   $\cap {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(mux(S_G[\![G_1]\!], S_G[\![G_4]\!]))$

   $\cap {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(mux(S_G[\![G_1]\!], S_G[\![G_2, G_3]\!]))$


1a Show: $\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G_1]\!]\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \leq 1$

$\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow \Theta \subseteq {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\mathcal{D}_G[\![G_1]\!])\delta_P$

hence (by (5) stated above): ${\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \subseteq \mathcal{D}_G[\![G_1]\!]\delta_P$

hence by assumption: $|\mathcal{F}_G[\![G_1]\!]\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \leq 1$


1b Show: $\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G_4]\!]\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \leq 1$

$\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow \Theta \subseteq {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\mathcal{D}_G[\![G_4]\!]\delta_P)$

hence (again by (5) above): ${\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \subseteq \mathcal{D}_G[\![G_4]\!]\delta_P$

hence by assumption: $|\mathcal{F}_G[\![G_4]\!]\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \leq 1$


1c Show: $\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G_3]\!]\mu_k[\Phi])| \leq 1$

where $\mathcal{F}_G[\![G_2]\!]\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = \Phi : \vec{\Phi}$

$\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow \Theta \subseteq {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(S_G[\![G_2]\!] \rightarrow \mathcal{D}_G[\![G_3]\!]\delta_P)$

hence (again by (5) above): ${\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \subseteq (S_G[\![G_2]\!] \rightarrow \mathcal{D}_G[\![G_3]\!]\delta_P)$

hence (by (1) stated above): ${\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_2]\!] \subseteq \mathcal{D}_G[\![G_3]\!]\delta_P$

by Theorem 1: $\bigcup(\Phi : \vec{\Phi}) = \bigcup(\mathcal{F}_G[\![G_2]\!]\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])) \subseteq {\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) \cap S_G[\![G_2]\!]$

therefore (since $\Phi \subseteq \bigcup(\Phi : \vec{\Phi})$): $\Phi \subseteq {\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) \cap S_G[\![G_2]\!] \subseteq \mathcal{D}_G[\![G_3]\!]\delta_P$

by assumption: $|\mathcal{F}_G[\![G_3]\!]\mu_k[\Phi]| \leq 1$


2a Show:

$\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow \neg(|\mathcal{F}_G[\![G_1]\!]\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \neq 0 \wedge |\mathcal{F}_G[\![G_4]\!]\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \neq 0)$

$\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow \Theta \subseteq {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(mux(S_G[\![G_1]\!], S_G[\![G_4]\!]))$

hence (by (5) stated above): ${\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \subseteq mux(S_G[\![G_1]\!], S_G[\![G_4]\!])$

hence (by (2) stated above):
$(\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_1]\!] = \{false\}) \vee (\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_4]\!] = \{false\})$
by Theorem 1: $\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_1]\!] = \{false\} \Rightarrow \mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = [\,]$
hence: $\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_1]\!] = \{false\} \Rightarrow |\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| = 0$
similarly: $\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_4]\!] = \{false\} \Rightarrow \mathcal{F}_G[\![G_4]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = [\,]$
hence: $\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_4]\!] = \{false\} \Rightarrow |\mathcal{F}_G[\![G_4]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| = 0$
therefore: $(|\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| = 0) \vee (|\mathcal{F}_G[\![G_4]\!]\mu_k\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| = 0)$
hence: $\neg((|\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \neq 0) \wedge (|\mathcal{F}_G[\![G_4]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \neq 0))$

2b Show: $\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow \neg(|\mathcal{F}_G[\![G_1]\!]\mu_k\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \neq 0 \wedge |\mathcal{F}_G[\![G_3]\!]\mu_k[\Phi]| \neq 0)$
where $\mathcal{F}_G[\![G_2]\!]\mu_k\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = \Phi : \vec{\Phi}$
$\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow \Theta \subseteq \downarrow\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(mux(S_G[\![G_1]\!], S_G[\![G_2, G_3]\!]))$
hence (again by (5) above): $\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \subseteq mux(S_G[\![G_1]\!], S_G[\![G_2, G_3]\!])$
hence (again by (2) above):
$(\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_1]\!] = \{false\}) \vee (\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_2, G_3]\!] = \{false\})$
by Theorem 1: $\Phi \subseteq \bigcup(\mathcal{F}_G[\![G_2]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])) \subseteq \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_2]\!]$
by Theorem 1: $\bigcup(\mathcal{F}_G[\![G_3]\!]\mu_k [\Phi]) \subseteq \Phi \cap S_G[\![G_3]\!] \subseteq \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_2]\!] \cap S_G[\![G_3]\!]$
hence: $\bigcup(\mathcal{F}_G[\![G_3]\!]\mu_k[\Phi]) \subseteq \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_2, G_3]\!]$
hence: $\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_2, G_3]\!] = \{false\} \Rightarrow \mathcal{F}_G[\![G_3]\!]\mu_k[\Phi] = [\,]$
hence: $\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_2, G_3]\!] = \{false\} \Rightarrow |\mathcal{F}_G[\![G_3]\!]\mu_k[\Phi]| = 0$
also (by Theorem 1): $\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_1]\!] = \{false\} \Rightarrow \mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = [\,]$
hence: $\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_1]\!] = \{false\} \Rightarrow |\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| = 0$
hence: $(|\mathcal{F}_G[\![G_1]\!]\mu_k\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| = 0) \vee (|\mathcal{F}_G[\![G_3]\!]\mu_k [\Phi]| = 0)$
hence: $\neg((|\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \neq 0) \vee (|\mathcal{F}_G[\![G_3]\!]\mu_k[\Phi]| \neq 0))$

2.2 Induction Step:
$G = G1, G2 :$
Assume: $\Theta_1 \subseteq \mathcal{D}_G[\![G_1]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta_1]| \leq 1$
And: $\Theta_2 \subseteq \mathcal{D}_G[\![G_2]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G_2]\!]\mu_{k+1}[\Theta_2]| \leq 1$
Show: $\Theta \subseteq \mathcal{D}_G[\![G]\!] \Rightarrow |\mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta]| \leq 1$
where $\mu_{k+1} = \mathcal{F}_P[\![P]\!]\mu_k$

$\mathcal{D}_G[\![G]\!]\delta_P = (S_G[\![G_2]\!] \rightarrow \mathcal{D}_G[\![G_1]\!]\delta_P) \cap (S_G[\![G_1]\!] \rightarrow \mathcal{D}_G[\![G_2]\!]\delta_P)$
therefore if $\Theta \subseteq \mathcal{D}_G[\![G]\!]\delta_P$
    then $\Theta \subseteq (S_G[\![G_1]\!] \rightarrow \mathcal{D}_G[\![G_2]\!]\delta_P)$
    and hence $\Theta \cap S_G[\![G_1]\!] \subseteq \mathcal{D}_G[\![G_2]\!]\delta_P$
similarly if $\Theta \subseteq \mathcal{D}_G[\![G]\!]\delta_P$
    then $\Theta \subseteq (S_G[\![G_2]\!] \rightarrow \mathcal{D}_G[\![G_1]\!]\delta_P)$
    and hence $\Theta \cap S_G[\![G_2]\!] \subseteq \mathcal{D}_G[\![G_1]\!]\delta_P$
by Lemma 2: $\mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta] = \mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta \cap S_G[\![G]\!]]$
applying the definition of $\mathcal{F}_G$:
$\mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta \cap S_G[\![G]\!]] = \mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]])$
now notice that: $\Theta \cap S_G[\![G_1, G_2]\!]$
      $= \Theta \cap S_G[\![G_1]\!] \cap S_G[\![G_2]\!]$
      $\subseteq \Theta \cap S_G[\![G_2]\!]$
      $\subseteq \mathcal{D}_G[\![G_1]\!]\delta_P$
hence by assumption: $|\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]| \leq 1$

distinguish two cases:
(a) $|\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]| = 0$,
(b) $|\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]| = 1$

(a) $|\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]| = 0$
$\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]] = [\,]$
$\mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]] = \mathcal{F}_G[\![G_2]\!]\mu_{k+1}[\,] = [\,]$
hence: $|\mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]| \leq 1$
hence by Lemma 2 (remembering $G = G_1, G_2$): $|\mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta]| \leq 1$

(b) $|\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]| = 1$
$\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]] = [\Psi]$
therefore: $\bigcup(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]) = \Psi$
by Theorem 1: $\Psi \subseteq \Theta \cap S_G[\![G_1, G_2]\!] \cap S_G[\![G_1]\!] \subseteq \Theta \cap S_G[\![G_1]\!]$
hence since $\Theta \cap S_G[\![G_1]\!] \subseteq \mathcal{D}_G[\![G_2]\!]\delta_P$ (see above): $\Psi \subseteq \mathcal{D}_G[\![G_2]\!]\delta_P$
hence by assumption: $|\mathcal{F}_G[\![G_2]\!]\mu_{k+1}[\Psi]| \leq 1$
hence (again using Lemma 2): $|\mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]])|$
$\quad = |\mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]|$
$\quad = |\mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta]| \leq 1$
QED

## 2.5  Abstraction Proofs

### 2.5.1  Proposition 1: If $\Theta_1 \subseteq \gamma_{\vec{x}}(f_1)$ and $\gamma_{\vec{x}}(f_1) \subseteq \Theta_2$ then $\gamma_{\vec{x}}(f_1 \Rightarrow f_2) \subseteq \Theta_1 \rightarrow \Theta_2$

$\gamma_{\vec{x}}(f_1 \Rightarrow f_2)$
$= \bigcup\{\gamma_{\vec{x}}(f) \mid f \models f_1 \Rightarrow f_2\}$
$= \bigcup\{\Theta \mid \alpha_{\vec{x}}(\Theta) \models f_1 \Rightarrow f_2\}$
$= \bigcup\{\Theta \mid (\alpha_{\vec{x}}(\Theta) \models f_1) \Rightarrow (\alpha_{\vec{x}}(\Theta) \models f_2)\}$
$= \bigcup\{\Theta \mid (\Theta \subseteq \gamma_{\vec{x}}(f_1)) \Rightarrow (\Theta \subseteq \gamma_{\vec{x}}(f_2))\}$
$\subseteq \bigcup\{\Theta \mid (\Theta \subseteq \Theta_1) \Rightarrow (\Theta \subseteq \Theta_2)\}$
$= \bigcup\{\Theta \mid (\Theta \subseteq \Theta_1 \cap \Theta_2) \vee (\Theta \not\subseteq \Theta_1)\}$
$= \bigcup\{\Theta \mid \Theta \subseteq (\Theta_1 \cap \Theta_2) \cup (Con \setminus \Theta_1)\}$
$= \bigcup\{\Theta \mid \Theta \cap \Theta_1 \subseteq \Theta_2\}$
$= \Theta_1 \rightarrow \Theta_2$

### 2.5.2  Proposition 2: $\gamma_{\vec{x}}(mux_{\vec{x}}^{\alpha}(\Theta_1^{DK}, \Theta_2^{DK})) \subseteq mux(\Theta_1, \Theta_2)$

Proof:
First notice that by the definition of the Galois connection (i.e. of $\gamma()$ and $\alpha()$ the following:
$\gamma_{\vec{x}}(mux_{\vec{x}}^{\alpha}(\Theta_1^{DK}, \Theta_2^{DK})) \subseteq mux(\Theta_1, \Theta_2)$
is equivalent to: $\alpha_{\vec{x}}(\Psi) \models mux_{\vec{x}}^{\alpha}(\Theta_1^{DK}, \Theta_2^{DK}) \rightarrow \Psi \subseteq mux(\Theta_1, \Theta_2)$

Now: $\alpha_{\vec{x}}(\Psi) \models mux_{\vec{x}}^{\alpha}(\Theta_1^{DK}, \Theta_2^{DK})$ iff for each clause in $\alpha_{\vec{x}}(\Psi)$ there is a clause in $mux_{\vec{x}}^{\alpha}(\Theta_1^{DK}, \Theta_2^{DK})$
that is entailed by it, ie:
$\forall \psi \in \Psi.\exists Y \subseteq vars(\vec{x}).(\forall \theta_1 \in \Theta_1^{DK}.\forall \theta_2 \in \Theta_2^{DK}.$

$$(\overline{\exists}_Y(\theta_1) \wedge \overline{\exists}_Y(\theta_2) = \textit{false}) \wedge \alpha_{\vec{x}}(\psi) \models \bigwedge Y)$$

Since $\textit{mux}_{\vec{x}}^{\alpha}(\Theta_1^{DK}, \Theta_2^{DK})$ contains only positive (ie non-negated) literals, only the positive literals entailed by $\alpha_{\vec{x}}(\psi)$ are relevant.

Now, the positive literals entailed by $\alpha_{\vec{x}}(\psi)$ are exactly $\textit{vars}(\vec{x}) \cap \textit{fix}(\psi)$.

Therefore: $\psi \in \gamma_{\vec{x}}(\textit{mux}_{\vec{x}}^{\alpha}(\Theta_1^{DK}, \Theta_2^{DK}))$

$\quad$ iff $\exists Y \subseteq (\textit{vars}(\vec{x}) \cap \textit{fix}(\psi)).(\forall \theta_1 \in \Theta_1^{DK}. \forall \theta_2 \in \Theta_2^{DK}.(\overline{\exists}_Y(\theta_1) \wedge \overline{\exists}_Y(\theta_2) = \textit{false}))$

Now observe that the following three things hold:

(1) $\forall \phi \in \Phi. \exists \phi' \in \Phi^{DK}.(\phi \models \phi')$

(2) $((f_1 \models f_1') \wedge (f_2 \models f_2') \wedge (f_1' \wedge f_2' = \textit{false})) \rightarrow f_1 \wedge f_2 = \textit{false}$

(3) $\phi \models \phi' \rightarrow \overline{\exists}_Y(\phi) \models \overline{\exists}_Y(\phi')$

Therefore from $\forall \theta_1' \in \Theta_1^{DK}. \forall \theta_2' \in \Theta_2^{DK}.(\overline{\exists}_Y(\theta_1') \wedge \overline{\exists}_Y(\theta_2') = \textit{false})$

it follows: $\forall \theta_1 \in \Theta_1. \forall \theta_2 \in \Theta_2.(\overline{\exists}_Y(\theta_1) \wedge \overline{\exists}_Y(\theta_2) = \textit{false})$

And thus: $\overline{\exists}_Y(\Theta_1) \cap \overline{\exists}_Y(\Theta_2) = \{\textit{false}\}$

Hence the following entailment holds:

$\forall \phi.(\exists Y \subseteq (\textit{vars}(\vec{x}) \cap \textit{fix}(\psi)).(\forall \theta_1 \in \Theta_1^{DK}. \forall \theta_2 \in \Theta_2^{DK}.(\overline{\exists}_Y(\theta_1) \wedge \overline{\exists}_Y(\theta_2) = \textit{false}))$

$\models$

$\quad \exists Y \subseteq \textit{fix}(\phi).(\overline{\exists}_Y(\Theta_1) \cap \overline{\exists}_Y(\Theta_2) = \{\textit{false}\}))$

Therefore: $\forall \phi.(\phi \in \gamma_{\vec{x}}(\textit{mux}_{\vec{x}}^{\alpha}(\Theta_1^{DK}, \Theta_2^{DK})) \rightarrow \phi \in \textit{mux}(\Theta_1, \Theta_2))$

From which it follows: $\gamma_{\vec{x}}(\textit{mux}_{\vec{x}}^{\alpha}(\Theta_1^{DK}, \Theta_2^{DK})) \subseteq \textit{mux}(\Theta_1, \Theta_2)$

## 2.6 Theorem 3: $\forall i \in \mathbb{N} : \gamma_{\textit{vars}(G)}(\mathcal{D}_G^{\alpha}[\![G]\!]\delta_i^{\alpha}) \subseteq \mathcal{D}_G[\![G]\!]\delta_i$ where $\delta_i^{\alpha}/\delta_i$ are the results of $i$ applications of $\mathcal{D}_P^{\alpha}[\![P]\!]/\mathcal{D}_P[\![P]\!]$ to $\delta_{\top}^{\alpha}/\delta_{\top}$ respectively.

Proof by nested induction on:

1. $i$,

2. the structure of $G$:

notice first that: $\gamma_{\textit{vars}(\vec{x})}(\rho_{\vec{y},\vec{x}}^{\alpha}\overline{\forall}_{\vec{y}}^{\alpha}(f)) \subseteq \rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\gamma_{\textit{vars}(\vec{y})}(f))$

1 Base Case: $i = 0$

$\delta_0^{\alpha} = \delta_{\top}^{\alpha}$

$\delta_0 = \delta_{\top}$

Show: $\gamma_{\textit{vars}(G)}(\mathcal{D}_G^{\alpha}[\![G]\!]\delta_{\top}^{\alpha}) \subseteq \mathcal{D}_G[\![G]\!]\delta_{\top}$

Induction on structure of $G$:

1.1 Two base cases: (1) $G = \textit{post}(\phi)$, (2) $G = p(\vec{x})$

(1) $G = \textit{post}(\phi)$

$\gamma_{\textit{vars}(\phi)}(\mathcal{D}_G^{\alpha}[\![\textit{post}(\phi)]\!]\delta_{\top}^{\alpha})$

$= \gamma_{\textit{vars}(\phi)}(\textit{true})$

$= \downarrow\!\{\textit{true}\}$

$= \mathcal{D}_G[\![\textit{post}(\phi)]\!]\delta_{\top}$

hence: $\gamma_{\textit{vars}(\phi)}(\mathcal{D}_G^{\alpha}[\![\textit{post}(\phi)]\!]\delta_{\top}^{\alpha}) \subseteq \mathcal{D}_G[\![\textit{post}(\phi)]\!]\delta_{\top}$

(2) $G = p(\vec{x})$

$\gamma_{\textit{vars}(\vec{x})}(\mathcal{D}_G^{\alpha}[\![p(\vec{x})]\!]\delta_{\top}^{\alpha})$

$= \gamma_{\textit{vars}(\vec{x})}(\rho_{\vec{y},\vec{x}}^{\alpha}\overline{\forall}_{\vec{y}}^{\alpha}(\textit{true}))$

$\subseteq \rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\gamma_{\textit{vars}(\vec{y})}(\textit{true}))$

$$= \rho_{\vec{y},\vec{x}} \overline{\forall}_{\vec{y}} (\Downarrow\{true\})$$
$$= \mathcal{D}_G[\![p(\vec{x})]\!]\delta_\top$$

1.2 Induction step: $G = G_1, G_2$
Assume: $\gamma_{vars(G_{1/2})}(\mathcal{D}_G^\alpha[\![G_{1/2}]\!]\delta_\top^\alpha) \subseteq \mathcal{D}_G[\![G_{1/2}]\!]\delta_\top$

$$\gamma_{vars(G_1,G_2)}(\mathcal{D}_G^\alpha[\![G_1, G_2]\!]\delta_\top^\alpha)$$
$$= \gamma_{vars(G_1,G_2)}((S_G^\alpha[\![G_2]\!] \Rightarrow \mathcal{D}_G^\alpha[\![G_1]\!]\delta_\top^\alpha) \wedge (S_G^\alpha[\![G_1]\!] \Rightarrow \mathcal{D}_G^\alpha[\![G_2]\!]\delta_\top^\alpha))$$
$$\subseteq \gamma_{vars(G_1,G_2)}(S_G^\alpha[\![G_2]\!] \Rightarrow \mathcal{D}_G^\alpha[\![G_1]\!]\delta_\top^\alpha) \cap \gamma_{vars(G_1,G_2)}(S_G^\alpha[\![G_1]\!] \Rightarrow \mathcal{D}_G^\alpha[\![G_2]\!]\delta_\top^\alpha)$$
$$\quad \text{(by monotonicity i.e. } \gamma_{vars(G_1,G_2)}(f_1 \wedge f_2) \subseteq \gamma_{vars(G_1,G_2)}(f_i))$$
$$\subseteq (S_G[\![G_2]\!] \to \mathcal{D}_G[\![G_1]\!]\delta_\top) \cap (S_G[\![G_1]\!] \to \mathcal{D}_G[\![G_2]\!]\delta_\top)$$
$$\quad \text{(by Proposition 1 and Proposition 3 and the induction assumption)}$$
$$= \mathcal{D}_G[\![G_1, G_2]\!]\delta_\top$$

2 Induction step: $i = k+1$
Assume: $\gamma_{vars(G)}(\mathcal{D}_G^\alpha[\![G]\!]\delta_k^\alpha) \subseteq \mathcal{D}_G[\![G]\!]\delta_k$
Show: $\gamma_{vars(G)}(\mathcal{D}_G^\alpha[\![G]\!]\delta_{k+1}^\alpha) \subseteq \mathcal{D}_G[\![G]\!]\delta_{k+1}$
where $\delta_{k+1} = \mathcal{D}_P[\![P]\!]\delta_k$ and $\delta_{k+1}^\alpha = \mathcal{D}_P^\alpha[\![P]\!]\delta_k^\alpha$

Induction on structure of G:
2.1 Two base cases: (1) $G = post(\phi)$, (2) $G = p(\vec{x})$
(1) $G = post(\phi)$
$$\gamma_{vars(\phi)}(\mathcal{D}_G^\alpha[\![post(\phi)]\!]\delta_{k+1}^\alpha)$$
$$= \gamma_{vars(\phi)}(true)$$
$$= \Downarrow\{true\}$$
$$= \mathcal{D}_G[\![post(\phi)]\!]\delta_{k+1}$$
hence: $\gamma_{vars(\phi)}(\mathcal{D}_G^\alpha[\![post(\phi)]\!]\delta_\top^\alpha) \subseteq \mathcal{D}_G[\![post(\phi)]\!]\delta_\top$

(2) $G = p(\vec{x})$
Assume (without loss of generality): $p(\vec{y}) \leftarrow G_1; G_2, !, G_3; G_4 \in P$

$$\gamma_{vars(\vec{x})}(\mathcal{D}_G^\alpha[\![p(\vec{x})]\!]\delta_{k+1}^\alpha)$$
$$= \gamma_{vars(\vec{x})}(\rho_{\vec{y},\vec{x}}^\alpha(\overline{\forall}_{\vec{y}}^\alpha(\mathcal{D}_G^\alpha[\![p(\vec{y})]\!]\delta_{k+1}^\alpha)))$$
$$= \gamma_{vars(\vec{x})}(\rho_{\vec{y},\vec{x}}^\alpha(\overline{\forall}_{\vec{y}}^\alpha(\overline{\forall}_{\vec{y}}^\alpha(\mathcal{D}_G^\alpha[\![G_1]\!]\delta_k^\alpha \wedge (S_G^\alpha[\![G_2]\!] \Rightarrow \mathcal{D}_G^\alpha[\![G_3]\!]\delta_k^\alpha) \wedge \mathcal{D}_G^\alpha[\![G_4]\!]\delta_k^\alpha$$
$$\quad \wedge mux_{vars(\vec{y})}^\alpha(S_G^{DK}[\![G_1]\!], S_G^{DK}[\![G_4]\!])$$
$$\quad \wedge mux_{vars(\vec{y})}^\alpha(S_G^{DK}[\![G_1]\!], S_G^{DK}[\![G_2, G_3]\!])))))$$
$$= \gamma_{vars(\vec{x})}(\rho_{\vec{y},\vec{x}}^\alpha(\overline{\forall}_{\vec{y}}^\alpha(\mathcal{D}_G^\alpha[\![G_1]\!]\delta_k^\alpha \wedge (S_G^\alpha[\![G_2]\!] \Rightarrow \mathcal{D}_G^\alpha[\![G_3]\!]\delta_k^\alpha) \wedge \mathcal{D}_G^\alpha[\![G_4]\!]\delta_k^\alpha$$
$$\quad \wedge mux_{vars(\vec{y})}^\alpha(S_G^{DK}[\![G_1]\!], S_G^{DK}[\![G_4]\!])$$
$$\quad \wedge mux_{vars(\vec{y})}^\alpha(S_G^{DK}[\![G_1]\!], S_G^{DK}[\![G_2, G_3]\!]))))$$
$$\subseteq \rho_{\vec{y},\vec{x}} \overline{\forall}_{\vec{y}} (\gamma_{vars(\vec{y})}(\mathcal{D}_G^\alpha[\![G_1]\!]\delta_k^\alpha \wedge (S_G^\alpha[\![G_2]\!] \Rightarrow \mathcal{D}_G^\alpha[\![G_3]\!]\delta_k^\alpha) \wedge \mathcal{D}_G^\alpha[\![G_4]\!]\delta_k^\alpha$$
$$\quad \wedge mux_{vars(\vec{y})}^\alpha(S_G^{DK}[\![G_1]\!], S_G^{DK}[\![G_4]\!])$$
$$\quad \wedge mux_{vars(\vec{y})}^\alpha(S_G^{DK}[\![G_1]\!], S_G^{DK}[\![G_2, G_3]\!])))$$
$$\subseteq \rho_{\vec{y},\vec{x}} \overline{\forall}_{\vec{y}} (\gamma_{vars(\vec{y})}(\mathcal{D}_G^\alpha[\![G_1]\!]\delta_k^\alpha) \cap (\gamma_{vars(\vec{y})}(S_G^\alpha[\![G_2]\!]) \to \gamma_{vars(\vec{y})}(\mathcal{D}_G^\alpha[\![G_3]\!]\delta_k^\alpha)) \cap \gamma_{vars(\vec{y})}(\mathcal{D}_G^\alpha[\![G_4]\!]\delta_k^\alpha)$$
$$\quad \cap \gamma_{vars(\vec{y})}(mux_{vars(\vec{y})}^\alpha(S_G^{DK}[\![G_1]\!], S_G^{DK}[\![G_4]\!])))$$
$$\quad \cap \gamma_{vars(\vec{y})}(mux_{vars(\vec{y})}^\alpha(S_G^{DK}[\![G_1]\!], S_G^{DK}[\![G_2, G_3]\!])))$$

$$\subseteq \rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\mathcal{D}_G[\![G_1]\!]\delta_k \cap (S_G[\![G_2]\!] \to \mathcal{D}_G[\![G_3]\!]\delta_k) \cap \mathcal{D}_G[\![G_4]\!]\delta_k$$
$$\cap \; mux(S_G[\![G_1]\!], S_G[\![G_4]\!])$$
$$\cap \; mux(S_G[\![G_1]\!], S_G[\![G_2, G_3]\!]))$$
$$= \mathcal{D}_G[\![p(\vec{x})]\!]\delta_{k+1}$$

2.2 Induction step: $G = G_1, G_2$
Assume: $\gamma_{vars(G_{1/2})}(\mathcal{D}_G^{\alpha}[\![G_{1/2}]\!]) \subseteq \mathcal{D}_G[\![G_{1/2}]\!]$
again, notice that: (1) $A \subseteq B \Rightarrow \gamma_B(f) \subseteq \gamma_A(f)$
and: $vars(G_1, G_2) = vars(G_1) \cup vars(G_2)$
and hence: $(2^1)$ $vars(G_1) \subseteq vars(G_1, G_2)$
and similarly: $(2^2)$ $vars(G_2) \subseteq vars(G_1, G_2)$

$$\gamma_{vars(G_1,G_2)}(\mathcal{D}_G^{\alpha}[\![G_1, G_2]\!]\delta_{k+1}^{\alpha})$$
$$= \gamma_{vars(G_1,G_2)}((S_G^{\alpha}[\![G_2]\!] \Rightarrow \mathcal{D}_G^{\alpha}[\![G_1]\!]\delta_{k+1}^{\alpha}) \wedge (S_G^{\alpha}[\![G_1]\!] \Rightarrow \mathcal{D}_G^{\alpha}[\![G_2]\!]\delta_{k+1}^{\alpha}))$$
$$\subseteq \gamma_{vars(G_1,G_2)}((S_G^{\alpha}[\![G_2]\!] \Rightarrow \mathcal{D}_G^{\alpha}[\![G_1]\!]\delta_{k+1}^{\alpha})) \cap \gamma_{vars(G_1,G_2)}((S_G^{\alpha}[\![G_1]\!] \Rightarrow \mathcal{D}_G^{\alpha}[\![G_2]\!]\delta_{k+1}^{\alpha}))$$
(by monotonicity i.e. $\gamma_{vars(G_1,G_2)}(f_1 \wedge f_2) \subseteq \gamma_{vars(G_1,G_2)}(f_i)$)
$$\subseteq \gamma_{vars(G_1,G_2)}(S_G^{\alpha}[\![G_2]\!]) \to \gamma_{vars(G_1,G_2)}(\mathcal{D}_G^{\alpha}[\![G_1]\!]\delta_{k+1}^{\alpha}) \cap \gamma_{vars(G_1,G_2)}(S_G^{\alpha}[\![G_1]\!]) \to \gamma_{vars(G_1,G_2)}(\mathcal{D}_G^{\alpha}[\![G_2]\!]\delta_{k+1}^{\alpha})$$
(by Proposition 1 and Proposition 3 and the induction assumption)
$$\subseteq \gamma_{vars(G_2)}(S_G^{\alpha}[\![G_2]\!]) \to \gamma_{vars(G_1)}(\mathcal{D}_G^{\alpha}[\![G_1]\!]\delta_{k+1}^{\alpha}) \cap \gamma_{vars(G_1)}(S_G^{\alpha}[\![G_1]\!]) \to \gamma_{vars(G_2)}(\mathcal{D}_G^{\alpha}[\![G_2]\!]\delta_{k+1}^{\alpha})$$
(by (1), $(2^1)$ and $(2^2)$ above)
$$\subseteq S_G[\![G_2]\!] \to \mathcal{D}_G[\![G_1]\!]\delta_{k+1} \cap S_G[\![G_1]\!] \to \mathcal{D}_G[\![G_2]\!]\delta_{k+1}$$
$$= \mathcal{D}_G[\![G_1, G_2]\!]\delta_{k+1}$$
QED

# References

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