A SOUND AND COMPLETE DIAGRAMMATIC REASONING SYSTEM

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ABSTRACT

Simple spider diagrams are a system of Venn-type diagrams that can be used to reason diagrammatically about sets, their cardinalities and their relationships. They extend the systems of Venn-Peirce diagrams investigated by Shin to include lower bounds for the cardinalities of the sets represented by the diagrams. This paper summarises the main syntax and semantics of simple spider diagrams and introduces inference rules for reasoning with the system. We discuss the soundness and prove completeness of the system. In proving completeness, we develop a proof strategy that is simpler than that adopted by Shin. We expect this strategy to extend to other, richer diagrammatic systems including other spider and constraint diagram systems. The general aim of this work is to provide the necessary mathematical underpinning for the development of software tools to aid reasoning with diagrams.

Keywords Diagrammatic reasoning, visual formalisms.

1. INTRODUCTION

In OO software development, diagrammatic modelling notations are used to specify systems. Recently, the Unified Modeling Language (UML) [10] has become the Object Management Group's (OMG) standard for such notations. In UML, constraints are expressed using the Object Constraint Language (OCL), essentially a stylised, textual, form of first-order predicate logic, which is part of the UML standard. *Constraint diagrams* [4, 8] are a diagrammatic notation for expressing constraints and can be used in conjunction with UML and OCL. *Spider diagrams* [3, 7] emerged from work on constraint diagrams. They combine and extend Venn diagrams and Euler circles to express constraints on sets and their relationships with other sets.

The notation we now call Euler circles was introduced by Euler in 1761 to illustrate relations between classes [2]. In 1880, Venn modified this notation to represent logical propositions [13]. In the 1890s, Peirce modified Venn diagrams to introduce elements and disjunctive information into the system [9]. Recently, full formal semantics and inference rules have been developed for Venn-Peirce diagrams [12] and Euler diagrams [6]; see also [1, 5] for related work.

This paper extends these diagrammatic inference rules to *simple spider diagrams*. Simple spider diagrams, in effect, enhance the semantics of the second Venn-Pierce system that Shin investigated (i.e., Venn-II, see [12] Chapter 4) to give lower bounds for the cardinality of the sets represented by the diagrams. Shin's proof of completeness does not extend very easily to spider diagram or constraint diagram systems; the central notion of a *maximal diagram* is not easy to define for these systems. In proving completeness of the simple spider diagram system, we give a proof strategy that should be extensible to most spider/constraint diagram systems and other similar systems, and can be adapted to give a simpler proof of the completeness of the Venn-II system than the one given by Shin.

The general aim of this work is to provide the necessary mathematical underpinning for the development of software tools to aid reasoning with diagrams. Some steps have been taken towards the development of tools. On the syntactic level, a good editor for constructing constraint diagrams and spider diagrams has been developed at Technion, [14]. An experimental system 'JVenn' has been developed at Niigata University for reasoning with a limited system of Venn diagrams, but this work is at an early stage of development [11].

The structure of this paper is as follows. A discussion of simple spider diagrams is conducted in section 2, where the main syntax and semantics of the notation is introduced. Section 3 introduces the inference rules for reasoning with simple spider diagrams and for combining diagrams. Section 4 considers the validity of the inference rules culminating in the soundness theorem. Section 5 gives the strategy for proving completeness and proves the completeness theorem. Section 6 states the conclusions of this paper and details related, ongoing and future work. Throughout this paper, for space reasons, some details of proofs, and sometimes whole proofs, will be omitted; however, sufficient information will be given to judge the veracity of the approach.

2. SIMPLE SPIDER DIAGRAMS

This section introduces the main syntax and semantics of simple spider diagrams. Spider diagrams, introduced in [3, 7], contain other syntactic elements which enable the expression of relations between elements and are based on

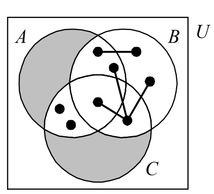
Euler diagrams rather than Venn diagrams. Simple spider diagrams considered here have a reduced set of syntactic components than spider diagrams; they are Venn-Peirce diagrams adapted so that we can infer lower bounds for the cardinalities of the sets represented by the non-empty regions.

2.1. Syntactic elements of unitary simple spider diagrams

A *contour* is a simple closed plane curve. A *boundary rectangle* properly contains all other contours. A *district* (or *basic region*) is the bounded subset of the plane enclosed by a contour. A *region* is defined, recursively, as follows: any district is a region; if r_1 and r_2 are regions, then the union, intersection, or difference, of r_1 and r_2 are regions provided these are non-empty. A *zone* (or *minimal region*) is a region having no other region contained within it. Contours and regions denote sets.

A *spider* is a tree with nodes (called *feet*) placed in different zones; the connecting edges (called *legs*) are straight lines. A spider *touches* a zone if one of its feet appears in that region. A spider may touch a zone at most once. A spider is said to *inhabit* the region which is the union of the zones it touches. For any spider s, the *habitat* of s, denoted $\eta(s)$, is the region inhabited by s. The set of complete spiders within region r is denoted by $S(r)$. A spider denotes the existence of an element in the set denoted by the habitat of the spider. Two distinct spiders denote distinct elements.

Every region is a union of zones. A region is *shaded* if each of its component zones is shaded. A shaded region denotes the empty set. No spider's foot can touch a shaded region. A *unitary simple spider diagram* is a boundary rectangle together with a finite collection of contours (all possible intersections of contours must occur, i.e., the underlying diagram is a Venn diagram), spiders and shaded regions. Each contour must be labelled and no two contours in the same unitary diagram can have the same label. The labelling of spiders is optional. For any unitary simple spider diagram D, we use $C = C(D)$, $Z = Z(D)$, $Z^* = Z^*(D)$, $R = R(D)$, $R^* = R^*(D)$, $L = L(D)$ and $S = S(D)$ to denote the sets of contours, zones, shaded zones, regions, shaded regions, contour labels and spiders of D, respectively.



D

Figure 1

The diagram D in Figure 1 can be interpreted as:

$$A-(B\cup C)=\varnothing \;\wedge\; C-(A\cup B)=\varnothing \;\wedge\; |A\cap C-B|\geq 2 \;\wedge\; \exists x,y\bullet x\in B \;\wedge\; y\in B-C\wedge x\neq y.$$

2.2. Semantics of unitary simple spider diagrams

A *model* for a unitary simple spider diagram D is a pair $m = (\Psi, \mathbf{U})$ where $\mathbf{U}$ is a set and $\Psi : C \to \mathit{Set}\,\mathbf{U}$, where $\mathit{Set}\,\mathbf{U}$ denotes the power set of $\mathbf{U}$, is a function mapping contours to subsets of $\mathbf{U}$. The boundary rectangle is interpreted as $\mathbf{U}$.

A zone is uniquely defined by the contours containing it and the contours not containing it; its interpretation is the intersection of the sets denoted by the contours containing it and the complements of the sets denoted by those contours not containing it. We extend the domain of Ψ to interpret regions as subsets of $\mathbf{U}$. First we define $\Psi: Z \to \mathit{Set}\,\mathbf{U}$ by

$$\Psi(z) = \bigcap_{c\in C^+(z)} \Psi(c) \cap \bigcap_{c\in C^-(z)} \overline{\Psi(c)}$$

where $C^+(z)$ and is the set of contours containing the zone z, $C^-(z)$ is the set of contours not containing z and $\overline{\Psi(c)} = \mathbf{U} - \Psi(c)$. Since any region is a union of zones, we may define $\Psi: R \to \mathit{Set}\,\mathbf{U}$ by

$$\Psi(r) = \bigcup_{z\in Z(r)} \Psi(z)$$

where, for any region r, $Z(r)$ is the set of zones contained in r.

The semantics predicate $P_D(m)$ of a unitary diagram D is the conjunction of the following three conditions.

Spider Condition: A spider denotes the existence of an element in the set denoted by the habitat of the spider:

$$\bigwedge_{s\in S} \exists x_s \bullet x_s \in \Psi(\eta(s))$$

Distinct Spiders Condition: The elements denoted by two distinct spiders are distinct:

$$\bigwedge_{s,t\in S,\, s\neq t} \exists x_s, x_t \bullet x_s \neq x_t$$

Shading Condition: The set denoted by a shaded zone is empty:

$$\bigwedge_{z\in Z^*} \Psi(z)=\varnothing$$

Theorem 1 The cardinality of the set denoted by region r of unitary diagram D is greater than or equal to the number of complete spiders in r:

$$\forall r\in R(D)\bullet |\Psi(r)|\geq |S(r)|$$

Theorem 1 is equivalent to the conjunction of the Spider and Distinct Spider Conditions. The proof is omitted.

2.3. Compound diagrams and multi-diagrams

Given two unitary diagrams D_1 and D_2, we can *connect* D_1 and D_2 with a straight line to produce a diagram $D = D_1 – D_2$. If a diagram has one boundary rectangle, then it is a *unitary* diagram; if a diagram has more than one rectangle, then it is a *compound* diagram. If a compound diagram D has n components, then we can place those n components in any order. Hence, for example, $D_1–D_2 = D_2–D_1$.

The semantics predicate of a compound diagram D is the disjunction of the semantics predicates of its component unitary diagrams; the boundary rectangles of the component unitary diagrams are interpreted as the same set **U**. That is, if $D = D_1–D_2– \dots –D_n$ then

$$P_D(m) = \bigvee_{i=1}^{n} P_{D_i}(m)$$

Contours with the same labels in different component unitary diagrams of a compound diagram D are interpreted as the same set:

$$\forall c_1, c_2 \in C(D) \bullet \ell(c_1) = \ell(c_2) \Rightarrow \Psi(c_1) = \Psi(c_2)$$

where $\ell(c)$ is the label of contour c.

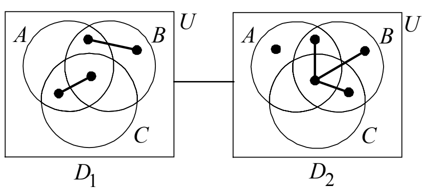


Figure 2

The compound diagram D in Figure 2 asserts that:

$$(\exists x, y \bullet x \in A \cap C \wedge y \in B - C)$$
$$\vee\ (\exists x, y \bullet x \in B \wedge y \in A - B \cup C).$$

A simple spider multi-diagram is a finite collection Δ of simple spider diagrams. The semantics predicate of a multi-diagram is the conjunction of the predicates of the individual diagrams; the boundary rectangles of all diagrams are interpreted as the same set **U**:

$$P_\Delta(m) = \bigwedge_{D \in \Delta} P_D(m) .$$

Contours with the same labels in different individual diagrams of a multi-diagram Δ are interpreted as the same set:

$$\forall c_1, c_2 \in \bigcup_{D_i \in \Delta} C(D_i) \bullet \ell(c_1) = \ell(c_2) \Rightarrow \Psi(c_1) = \Psi(c_2) .$$

2.4 Comparing regions across diagrams

Given two unitary diagrams D and D', we wish to call regions in the two diagrams ‘equivalent’ if they represent the same set. We can make this precise by considering partitions of the set of contour labels the two diagrams have in common. Let D be a unitary diagram. For any $z \in Z(D)$, define $L^+(z) = \ell(C^+(z))$, the set of labels of the contours containing z, and $L^-(z) = \ell(C^-(z))$, the set of labels of the contours not containing z. Let $P = (L^+, L^-)$ be a partition of $L(D) \cap L(D')$ and define $Z_P(D) = \{z \in Z(D) \mid L^+ = L^+(z) \cap L(D') \wedge L^- = L^-(z) \cap L(D')\}$. A region $zr \in R(D)$ is said to be *zonal with respect to* D' if there exists a partition P of $L(D) \cap L(D')$ such that

$$zr = \bigcup_{z \in Z_P(D)} z .$$

Suppose region zr of D is zonal with respect to D' and zr' of D' is zonal with respect to D. Then zr and zr' are *corresponding zonal regions*, denoted $zr \equiv_c zr'$, if there exists a partition P of $L(D) \cap L(D')$ such that

$$zr = \bigcup_{z \in Z_P(D)} z \quad \text{and} \quad zr' = \bigcup_{z' \in Z_P(D')} z' .$$

Let r be a region of D and let r' be a region of D'. Then r and r' are *corresponding regions*, denoted by $r' \equiv_c r$, if and only if r is a union of a set $ZR(r)$ of zonal regions with respect to D', r' is a union of a set $ZR(r')$ of zonal regions with respect to D, and

$$\left(\forall zr \in ZR(r)\ \exists zr' \in ZR(r') \bullet zr \equiv_c zr'\right)$$
$$\wedge \left(\forall zr' \in ZR(r')\ \exists zr \in ZR(r) \bullet zr \equiv_c zr'\right).$$

Theorem 2 The relation $\equiv_c$ is an equivalence relation.

If $r_1 \in R(D)$ and $r_1 \subseteq r \equiv_c r'$, then r_1 is a *corresponding subregion* of r', denoted by $r_1 \subseteq_c r'$.

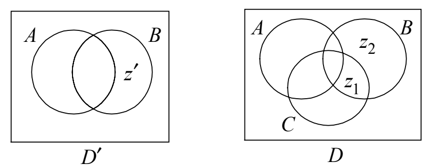


Figure 3

In Figure 3, the region $z = z_1 \cup z_2$ in D is zonal with respect to D' and the region z' in D' is zonal with respect to D. Furthermore, $z' \equiv_c z$ as both regions are associated with the partition $P = (\{B\}, \{A\})$ of $L(D) \cap L(D') = \{A, B\}$; hence $z_1 \subseteq_c z'$ and $z_2 \subseteq_c z'$.

The following theorem shows that the corresponding region relations behave well with respect to the semantics.

Theorem 3

(i) $\forall r \in R(D)\ \ \forall r' \in R(D')\ \ \forall m = (\Psi,\ \mathbf{U})\ \bullet$
$P_D(m) \wedge P_{D'}(m)\ \bullet\ \left(r \equiv_c r' \Rightarrow \Psi(r) = \Psi(r')\right).$

(ii) $\forall r \in R(D)\ \forall r' \in R(D')\ \forall m = (\Psi, \mathbf{U})\ \bullet$
$P_D(m) \wedge P_{D'}(m)\ \bullet\ \left(r \subseteq_c r' \Rightarrow \Psi(r) \subseteq \Psi(r')\right).$

The proof is omitted. We can now give a definition of equivalent diagrams. Two unitary diagrams D and D' are *equivalent*, denoted by $D \equiv D'$, if

(i) $L(D) = L(D')$,

(ii) $\forall r \in R^*(D)\ \exists r' \in R'^*(D') \bullet r \equiv_c r'$
$\wedge\ \forall r' \in R'^*(D')\ \exists r \in R^*(D) \bullet r \equiv_c r'$ and

(iii) $\forall r \in R(D)\, \forall r' \in R'(D') \bullet r \equiv_c r' \Rightarrow |S(r)| = |S(r')|$.

2.5. Satisfiability and Consistency

A model $m = (\Psi, \mathbf{U})$ *complies* with diagram D, $m \vDash D$, if it satisfies its semantic predicate $P_D(m)$. That is, $m \vDash D \Leftrightarrow P_D(m)$. Similarly, a model m *complies* with multi-diagram Δ if it satisfies its semantic predicate $P_\Delta(m)$. That is, $m \vDash \Delta \Leftrightarrow P_\Delta(m)$. A diagram is *satisfiable* if and only if it has a compliant model. Similarly, a multi-diagram is *satisfiable* if and only if it has a compliant model.

Theorem 4 All simple spider diagrams are satisfiable.

The proof is based on the construction of topological models for the diagram. The details are omitted. Theorem 4 does not extend to multi-diagrams.

The syntactic notion of consistency for pairs of diagrams is defined as follows:

(i) Two unitary simple spider diagrams D and D' are *consistent* if and only if

$$\forall r \in R(D) \;\; \forall r' \in R(D') \bullet r \equiv_c r' \Rightarrow$$
$$\neg(r \in R^*(D) \wedge |S(r')| > 0) \wedge \neg(r' \in R^*(D') \wedge |S(r)| > 0).$$

(ii) Two simple spider diagrams D and D' are *consistent* if and only if there exist unitary components D_i of D and D'_j of D' such that D_i and D'_j are consistent.

Intuitively, the condition (i) above would prevent the case in which a region is shaded in one diagram but the corresponding region in the other contains a spider; this is the only case in which a pair of unitary diagrams can be inconsistent.

Theorem 5 Let $\Delta = \{D_1, D_2\}$. Then Δ is satisfiable if and only if Δ is consistent.

Figure 4 shows a multi-diagram which is inconsistent, but whose components are all pairwise consistent. Discussion of the consistency of multi-diagrams in general is deferred until we consider combining diagrams.

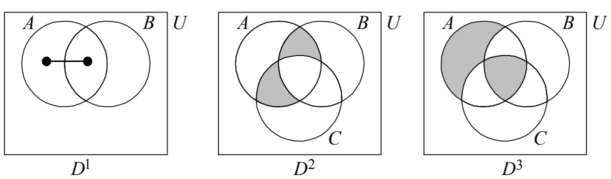


Figure 4

3. DIAGRAMMATIC REASONING RULES

We introduce purely syntactic, diagrammatic rules for turning one diagram into another. In this section we define and illustrate the rules; in the next section we show that the rules are valid.

3.1. Rules of transformation for unitary diagrams

We introduce rules that allow us to obtain one unitary diagram from a given unitary diagram by removing, adding or modifying diagrammatic elements. Each rule is either self-explanatory or is followed by a figure that illustrates it.

Rule 1: Erasure of shading. We may erase the shading in an entire zone.

Rule 2: Erasure of a spider. We may erase a complete spider.

Rule 3: Erasure of a contour. We may erase a contour. When a contour is erased:

- any shading remaining in only a part of a zone should also be erased.
- if a spider has feet in two regions which combine to form a single zone with the erasure of the contour, then these feet are replaced with a single foot connected to the rest of the spider.

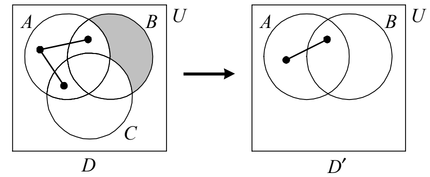


Figure 5

Rule 4: Spreading the feet of a spider. If a diagram has a spider s, then we may draw a node in any non-shaded zone which does not contain a foot of s and connect it to s.

Figure 6 illustrates rule 4. From D we know that there is an element belonging to $A - B$. Having spread its feet in D', we may only infer that this element belongs to A.

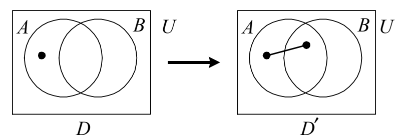


Figure 6

Rule 5: Introduction of a contour. A new contour may be drawn interior to the bounding rectangle observing the partial-overlapping rule: each zone splits into two zones with the introduction of the new contour. Each foot of a spider is replaced with a connected pair of feet, one in each new zone.

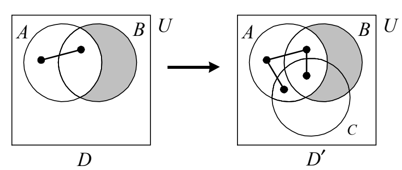


Figure 7

3.2. Rules of transformation involving compound diagrams

Rule 6: Splitting spiders. If a unitary diagram D has a spider s whose habitat is formed by n zones, then we may replace D with a connection of n unitary diagrams D_1– … – D_n where each foot of the spider s touches a different corresponding zone in each diagram D_i.

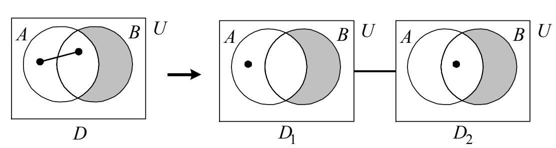


Figure 8

Rule 7: Excluded middle. If a unitary diagram D has a non-shaded zone z touched by no spiders, then we may replace D with D_1–D_2, where D_1 and D_2 are unitary and one of the corresponding zones of z is shaded and the other touched by a single foot spider.

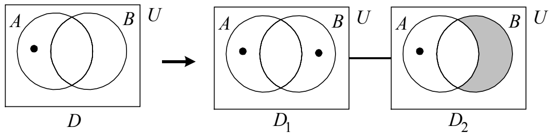


Figure 9

Rule 8: Connecting a diagram. For a given diagram D, we may connect any diagram D' to D.

Rule 9: Construction. Given a diagram $D_1 - \ldots - D_n$, we may transform it into D if each $D_1, \ldots, D_n$ may be transformed into D by some of the first eight transformation rules.

3.3 Combining Diagrams

Given two consistent diagrams, D^1 and D^2, we can combine them to produce a diagram D, losing no semantic information in the process. In this section we describe the construction of such a combined diagram D.

An α diagram is a diagram in which no spider's legs appear, that is, the habitat of any spider is a zone. Any diagram D can be transformed into an α diagram by repeated application of rule 6, splitting spiders. We give the definition of combining diagrams in several stages.

Let D^1 and D^2 be two consistent diagrams. Then their combination $D = D^1 * D^2$ is defined as follows.

(i) D^1 and D^2 are α unitary diagrams with $L(D^1) = L(D^2)$. The combined diagram D is also an α unitary diagram for which $L(D) = L(D^1) = L(D^2)$. So, for each $z \in Z(D)$, there exist equivalent zones $z_1 \in Z(D^1)$ and $z_2 \in Z(D^2)$. Furthermore, the number of spiders in z is equal to the maximum of the number of spiders in z_1 and the number of spiders in z_2, and z is shaded if and only if z_1 or z_2 is shaded.

$$\forall z \in Z(D)\, \exists z_1 \in Z(D^1)\, \exists z_2 \in Z(D^2) \bullet z \equiv_c z_1 \equiv_c z_2$$
$$\Rightarrow \; (|S(z)| = \max(|S(z_1)|, |S(z_2)|)$$
$$\wedge\, (z \in Z^*(D) \Leftrightarrow z_1 \in Z^*(D_1) \vee z_2 \in Z^*(D_2))$$

(ii) D^1 and D^2 are unitary diagrams and $L(D^1) \neq L(D^2)$. We introduce contours into D^1 and D^2 to obtain D^{1a} and D^{2b}, where $L(D^{1a}) = L(D^{2b}) = L(D^1) \cup L(D^2)$. Transform D^{1a} and D^{2b} into their α diagrams $D_1{}^{1a} - \ldots - D_n{}^{1a}$ and $D_1{}^{2b} - \ldots - D_m{}^{2b}$. The combined diagram D is the compound diagram formed by combining each $D_i{}^{1a}$ with each $D_j{}^{2b}$; where the two components are inconsistent, we do not obtain a corresponding component in D.

(iii) D^1 and D^2 are any consistent simple spider diagrams. The combined diagram D is the compound diagram formed by combining each component $D_i{}^1$ of D^1 with each component $D_j{}^2$ of D^2.

Part (ii) of the definition is illustrated in Figure 10. First, contour C is added to D^2 to form D^{2a}, which is then transformed into an α diagram. We then combine the components of each diagram. Note that D^{1a} and $D_2{}^{2a}$ are inconsistent, as are D^{1a} and $D_3{}^{2a}$, so the resulting combined diagram is unitary.

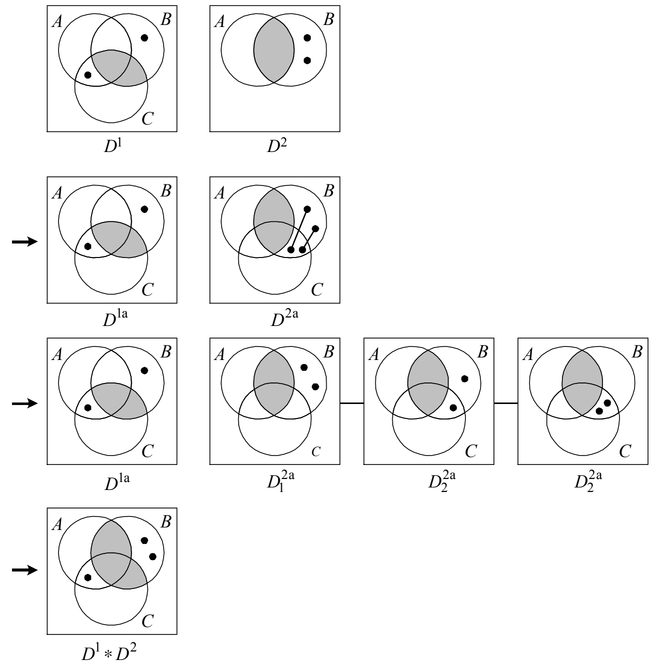


Figure 10

The combining operation $*$ is commutative and associative. This allows us to define the combination of the components of a multi-diagram $\Delta = \{D^1, D^2, \ldots, D^n\}$ unambiguously as $D^* = D^1 * D^2 * \ldots * D^n$. If Δ is inconsistent, then following the steps above will result in no diagram; D^* is only defined when Δ is consistent. A test for the consistency of Δ is to try to evaluate D^*.

Rule 10: Inconsistency. Given an inconsistent multi-diagram Δ, we may replace Δ with any multi-diagram.

Rule 11: Combining diagrams. A consistent multi-diagram $\Delta = \{D^1, D^2, \ldots, D^n\}$ may be replaced by the combined diagram $D^* = D^1 * D^2 * \ldots * D^n$.

4. SOUNDNESS

We write $\Delta \vdash D'$ to denote that diagram D' is obtained from multi-diagram Δ by applying a sequence of transformations. We write $D \vdash D'$ to mean $\{D\} \vdash D'$, etc. A diagram D' is a consequence of D, denoted by $D \vDash D'$, if every compliant model for D is also a compliant model for D'. A rule is *valid*, if $D \vdash D' \Rightarrow D \vDash D'$.

For space reasons, we omit the proofs of the validity of the rules. The proofs for rules 1 – 10 are similar to those of the Venn-II system given in [12] and the proofs are fairly straightforward and are omitted for space reasons. Rules 5, 6, 7 and 11 do not lose any semantic information; this fact is useful for proving completeness in the next section. The following result follows by induction from the validity of the rules.

Theorem 6 Soundness Theorem Let Δ be a multi-diagram and D a diagram. Then $\Delta \vdash D \Rightarrow \Delta \vDash D$.

5. COMPLETENESS

To prove completeness we show that if diagram D' is a consequence of multi-diagram Δ, then Δ can be transformed into D' by a finite sequence of applications of the rules given in section 3. That is, $\Delta \vDash D' \Rightarrow \Delta \vdash D'$.

The basic strategy is to transform Δ and D' into diagrams D and D'', respectively, so that if D'' is a consequence of diagram D, then there are diagrammatic conditions which must hold between D and D'', and if these conditions hold, then D can be transformed syntactically into D''. We first show that a multi-diagram is a consequence of the combination of its component diagrams. That is, if $\Delta = \{D^1, D^2, \ldots, D^n\}$ then any model m compliant with $D^* = D^1 * \ldots * D^n$ is also compliant with Δ, that is, $D^* \vDash \Delta$. The proofs of Theorems 7, 8 and 9 are very informal.

Theorem 7 Let $\Delta = \{D^1, \ldots, D^n\}$ and $D^* = D^1 * \ldots * D^n$. Then $D^* \vDash \Delta$.

Proof Each step in obtaining D^* is reversible. Thus for each i, $D^* \vdash D^i$. So, by soundness, $D^* \vDash \Delta$.

Theorem 8 Let D^c be the diagram obtained from D by rule 5, introduction of a contour. Then $D^c \vDash D$.

Proof Introducing a contour splits each spider in two; erasing that contour will reunite the spiders and have no other effect. So, to transform D^c into D, we erase the contour introduced into D. Thus, $D^c \vdash D$. So $D^c \vDash D$ by soundness.

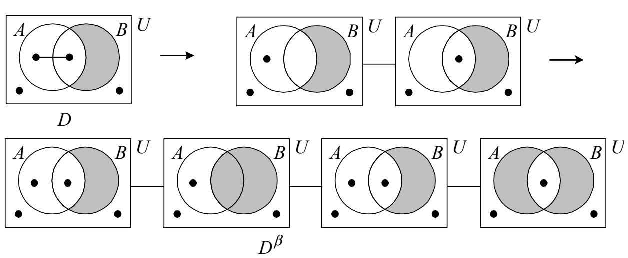


Figure 11

A β diagram is an α diagram in which each zone is either shaded or contains at least one spider. Any diagram D can be transformed into its β diagram D^β by repeated application of rule 6, splitting spiders, to turn it into an α diagram and then repeated application of rule 7, excluded middle. Figure 11 illustrates the transformation $D \vdash D^\beta$ for a unitary diagram D.

Theorem 9 For any diagram D, $D^\beta \vdash D$.

Any unitary component of D^β can be transformed into D by removing shading and extending spiders appropriately (i.e., by undoing the transformations to obtain that component). So, by rule 9, the rule of construction, $D^\beta \vdash D$.

Theorem 10 Let D and D' be β unitary diagrams for which $L(D') \subseteq L(D)$. Then the following three statements are equivalent.

(i) $D \vdash D'$

(ii) $D \vDash D'$

(iii) [1] $\forall z' \in Z^*(D')\ \exists r \in R^*(D) \bullet z' \subseteq_c r$ and

[2] $\forall r' \in R(D')\ \forall r \in R(D) \bullet r' \equiv_c r \Rightarrow |S(r)| \geq |S(r')|$

Proof (i) $\Rightarrow$ (ii). By soundness.

(ii) $\Rightarrow$ (iii). We will prove the contrapositive: $\neg([1] \wedge [2]) \Rightarrow \neg(D \vDash D')$. That is, $(\neg[1] \vee \neg[2]) \Rightarrow \neg(D \vDash D')$, which is equivalent to $\neg[1] \Rightarrow \neg(D \vDash D') \wedge \neg[2] \Rightarrow \neg(D \vDash D')$.

(a) $\neg[1] \Rightarrow \neg(D \vDash D')$. Assume $\neg[1]$; that is, $\exists z' \in Z^*(D')\ \forall r \in R^*(D) \bullet \neg(z' \subseteq_c r)$. Let z_1' be such a zone z'. Let $r_1 \in R(D)$ be such that $r_1 \equiv_c z_1'$ (r_1 exists because $L(D') \subseteq L(D)$). Then, by the assumption, r_1 is not entirely shaded and therefore contains spiders (as D is a β diagram). That is, $S(r_1) \neq \varnothing$. Then, for any model $m = (\Psi, \mathbf{U})$ compliant with D, $\Psi(r_1) > 0$. But no such m is compliant with D' since for any model $m = (\Psi, \mathbf{U})$ compliant with D', $\Psi(z_1') = \varnothing$, and by Theorem 3, $\Psi(z_1') = \Psi(r_1)$ as $z_1' \equiv_c r_1$.

(b) $\neg[2] \Rightarrow \neg(D \vDash D')$. Assume $\neg[2]$; that is, $\exists r' \in R(D')\ \exists r \in R(D) \bullet r' \equiv_c r \wedge |S(r)| < |S(r')|$. Let $r_1' \in R(D')$ and $r_1 \in R(D)$ be such that $r_1' \equiv_c r_1$ and $|S(r_1)| < |S(r_1')|$. Let $m = (\Psi, \mathbf{U})$ be such that $m \vDash D$ and $|\Psi(r_1)| = |S(r_1)|$. For any $m' = (\Psi', \mathbf{U})$ compliant with D' we have, $|\Psi'(r_1')| \geq |S(r_1')| > |S(r_1)| = |\Psi(r_1)|$, so m is not compliant with D'.

(iii) $\Rightarrow$ (i). Let D and D' be β unitary diagrams for which $L(D') \subseteq L(D)$ and assume [1] and [2]. Then any region in D' has a corresponding equivalent region in D. Let $r \in R(D)$ be such that $r \equiv_c \bigcup_{z \in Z^*(D')} z$, the region consisting of all and only shaded zones in D'. Erase the shading in $\bigcup_{z \in Z^*(D)} z - r$ to obtain D^1 so that $\bigcup_{z \in Z^*(D^1)} z \equiv_c \bigcup_{z \in Z^*(D')} z$. Remove contours from D^1 to obtain D^2 so that $L(D') = L(D^2)$. Delete spiders in D^2 to obtain D^3 so that $\forall r \in R(D)\ \forall r' \in R(D') \bullet r \equiv_c r' \Rightarrow |S(r)| = |S(r')|$. Then $D^3 \equiv D'$.

Theorem 11 Let $D, D_1, D_2, \ldots, D_n$ be β unitary diagrams for which $L(D_1) \cup L(D_2) \cup \ldots \cup L(D_n) \subseteq L(D)$. Then

$$D \vDash D_1 - D_2 - \ldots - D_n \Rightarrow (D \vDash D_1) \vee \ldots \vee (D \vDash D_n).$$

Proof We will prove the contrapositive. Assume $\forall D_i \bullet D \nvDash D_i$. By Theorem 10, either (i) $\exists z_i \in Z^*(D)$

$\forall r \in R^*(D) \bullet \neg(z_i \subseteq_c r)$ or (ii) $\exists r_i \in R(D_i)\ \exists r \in R(D) \bullet r \bullet\ r_i \equiv_c r \wedge |S(r)| < |S(r_i)|$. Let $m_1 = (\Psi, \mathbf{U})$ be compliant with D and be such that $\forall z \in Z(D) \bullet |\Psi_1(z)| = |S(z)|$.

If condition (i) holds in D_i, then there will be a shaded zone z_i in D_i for which the corresponding equivalent region r in D contains a spider. For any model $m = (\Psi, \mathbf{U})$ compliant with D_i, $\Psi(z_i) = \varnothing$ and by Theorem 2, $\Psi(z_i) = \Psi(r)$ as $z_i \equiv_c r$. So m_1 is not compliant with D_i.

If condition (ii) holds in D_i, then there will be a region r_i in D_i for which the corresponding equivalent region r in D contains fewer spiders. That is $r_i \equiv_c r \wedge |S(r)| < |S(r_i)|$. For any model $m = (\Psi, \mathbf{U})$ compliant with D_i, $|\Psi(r_i)| \geq |S(r_i)|$ and $\Psi(r_i) = \Psi(r)$. So m_1 is not compliant with D_i.

Therefore for $i = 1, 2, \ldots, n$, D_i, $m_1 \nvDash D_i$ and hence, $m_1 \nvDash D_1 - D_2 - \ldots - D_n$. So, there exists an m such that $m \vDash D$, but that $m \nvDash D_1 - D_2 - \ldots - D_n$. That is, $D \nvDash D_1 - D_2 - \ldots - D_n$.

Theorem 12 Let D and D' be compound diagrams for which each component is a β unitary diagram. That is, $D = D_1^{\beta} - \ldots - D_k^{\beta}$ and $D' = D'^{\beta}_1 - \ldots - D'^{\beta}_n$. Assume further that $\forall D_i^{\beta}\ \forall D'^{\beta}_j \bullet L(D'^{\beta}_j) \subseteq L(D_i^{\beta})$. Then $D \vDash D' \Rightarrow \forall D_i^{\beta}\ \exists D'^{\beta}_j \bullet D_i^{\beta} \vDash D'^{\beta}_j$.

Proof Assume $D \vDash D'$. Let $m = (\Psi, \mathbf{U})$ be any model such that $m \vDash D$. Then $m \vDash D'$. So $P_D(m) \Rightarrow P_{D'}(m)$. That is, $\bigvee_{i=1}^{k} P_{D_i^{\beta}}(m) \Rightarrow \bigvee_{i=1}^{n} P_{D'^{\beta}_i}(m)$. By logical manipulation we have $\forall D_i^{\beta} \bullet P_{D_i^{\beta}}(m) \Rightarrow \bigvee_{i=1}^{n} P_{D'^{\beta}_i}(m)$. That is, $\forall D_i^{\beta} \bullet D_i^{\beta} \vDash D'^{\beta}_1 - D'^{\beta}_2 - \cdots - D'^{\beta}_n$. Hence, by Theorem 11, $\forall D_i^{\beta}\ \exists D'^{\beta}_j \bullet D_i^{\beta} \vDash D'^{\beta}_j$.

Theorem 13 Completeness Theorem Let Δ be a multi-diagram and let D' be a diagram. Then $\Delta \vDash D' \Rightarrow \Delta \vdash D'$.

Proof If Δ is inconsistent, then the result follows immediately by applying Rule 10. Assume that Δ is consistent and that $\Delta \vDash D'$. By Theorem 7, $D^* \vDash \Delta$. So, by transitivity, $D^* \vDash D'$. Introduce contours into each component of D^* to produce $D^{*c} = D_1{}^{*c} - \ldots - D_m{}^{*c}$ so that $\forall D_i{}^{*c} \bullet L(D') \subseteq L(D_m{}^{*c})$. Then, by Theorem 8, $D^{*c} \vDash D^*$. So, by transitivity, $D^{*c} \vDash D'$. Transform D^{*c} and D' into their β diagrams, $D^{*c\beta}$ and D'^{β}, respectively. By Theorem 9 and the soundness theorem, $D^{*c\beta} \vDash D^{*c}$ and, by the soundness theorem, $D' \vDash D'^{\beta}$. So, by transitivity, $D^{*c\beta} \vDash D'^{\beta}$. Since $\forall D_i{}^{*c\beta}\ \forall D'^{\beta}_j \bullet L(D'^{\beta}_j) \subseteq L(D_i{}^{*c\beta})$ it follows from Theorem 12 that $\forall D_i{}^{*c\beta}\ \exists D'^{\beta}_j \bullet D_i{}^{*c\beta} \vDash D'^{\beta}_j$. Hence, by Theorem 10, $\forall D_i{}^{*c\beta}\ \exists D'^{\beta}_j \bullet D_i{}^{*c\beta} \vdash D'^{\beta}_j$. So, by applying rule 8, $\forall D_i{}^{*c\beta} \bullet D_i{}^{*c\beta} \vdash D'^{\beta}$ and hence, by applying rule 9, $D^{*c\beta} \vdash D'^{\beta}$. Now, $\Delta \vdash D^*$, $D^* \vdash D^{*c\beta}$, and, by Theorem 9, $D'^{\beta} \vdash D'$, so, by transitivity, $\Delta \vdash D'$.

6. CONCLUSION AND RELATED WORK

We have given formal syntax and semantics and diagrammatic inference rules to simple spider diagrams. We have shown that the inference rules are sound and complete. In proving completeness, we have provided a proof strategy that should be extensible to most spider/constraint diagram systems and other similar systems based on Venn or Euler diagrams. Indeed, the proof can be adapted to give a simpler proof of the completeness of the Venn-II system than the one given by Shin.

We are in the process of proving soundness and completeness of other spider diagram systems using the same strategy as that introduced in this paper. Our longer term aim is to prove similar results for constraint diagrams, and to provide the necessary mathematical underpinning for the development of software tools to aid the reasoning process.

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