SD2: A Sound and Complete Diagrammatic Reasoning System

John Howse, Fernando Molina and John Taylor
School of Computing & Mathematical Sciences
University of Brighton, UK
{John.Howse, F.Molina, John.Taylor}@bton.ac.uk

Abstract

SD2 is a system of Venn-type diagrams that can be used to reason diagrammatically about sets, their cardinalities and their relationships. They augment the systems of Venn-Peirce diagrams investigated by Shin to include lower and upper bounds for the cardinalities of the sets represented by regions of diagrams. This paper summarises their syntax and semantics and introduces inference rules for reasoning with the system. We discuss the soundness of the system and develop a proof strategy for completeness simpler than that adopted by Shin. We expect this strategy to extend to other, richer spider diagram systems and to constraint diagrams, the visual notation that has been used in conjunction with object-oriented modelling notations such as the Unified Modelling Language.

1. Introduction

Euler circles [2] are generally accepted as the first graphical notation for representing relations between classes and solving syllogisms. This notation is based on the correspondence between the topological properties of enclosure, exclusion and intersection and the set-theoretic notions of subset, disjoint sets, and set intersection, respectively. Venn [13] modified this notation to illustrate all possible relations between classes by showing all possible intersections of contours and by introducing shading in a region to denote the empty set. However a disadvantage of this system is its inability to represent existential statements. Peirce [10] modified and extended the Venn system by introducing the character 'x' to denote a non-empty set, the character 'o' to denote the empty set and a line between these marks to represent disjunctive information. Recently, full formal semantics and inference rules have been developed for Venn-Peirce diagrams [12] and Euler diagrams [6]; see also [1, 5] for related work. Shin proves soundness and completeness results for two systems of Venn-Peirce diagrams.

Spider diagrams [3, 7, 8] emerged from work on constraint diagrams [4, 9] and extend the system of Venn-Peirce diagrams investigated by Shin. Constraint diagrams are a visual diagrammatic notation for expressing constraints that can be used in conjunction with the Unified Modelling Language (UML) [11] and the Object Constraint Language (OCL) [14]. OCL is essentially a textual form of first-order predicate logic, which is part of the UML standard and used to express constraints, such as invariants, preconditions and postconditions.

In this paper we extend the diagrammatic rules and enhance the semantics of the second Venn-Peirce system that Shin investigated (i.e., Venn-II, see [12] Chapter 4) to express more information about the cardinality of represented sets. Shin introduced the notion of maximal diagram in the proof of completeness of her systems; the basic idea here is to construct a diagram that explicitly contains all the logical consequences of a given one. This approach is not easy to adapt to spider diagrams. We opt for a strategy in which the diagram that results from combining a set of diagrams and the diagram which is a consequence of that set are expanded in a way similar to disjunctive normal form in symbolic logic. This proof strategy should extend to most spider/constraint diagram systems.

A discussion of the system is conducted in section 2, where the main syntax and semantics of the notation is introduced. Section 3 introduces the inference rules for reasoning with spider diagrams and for combining diagrams and considers consistency and the validity of the inference rules. Section 4 gives the strategy for proving completeness and proves the completeness theorem. Section 5 states the conclusions of this paper and details related, ongoing and future work.

Throughout this paper, for space reasons, we omit most proofs and focus on the strategy for showing completeness of this and other spider diagram systems.

2. Spider Diagrams: SD2

This section introduces the main syntax and semantics of SD2, a subset of spider diagrams. In SD1 or *simple spider diagrams*, as defined in [7], we extended the diagrammatic rules and enhanced the semantics of Venn-II to give lower bounds for the cardinality of the sets represented by the diagrams and proved the soundness and completeness of the system. SD2 extends this system so that we can infer lower and upper bounds for the cardinalities of the sets represented by the diagrams. Spider diagrams contain other syntactic elements which enable the expression of relations between elements and form the basis of constraint diagrams, a rich notation allowing relations between sets to be expressed, see [3, 4, 8, 9] for more details.

2.1. Syntactic elements of unitary SD2 diagrams

A *contour* is a simple closed plane curve. A *boundary rectangle* properly contains all other contours. A *district* (or *basic region*) is the bounded *area* of the plane enclosed by a contour or by the boundary rectangle. A *region* is defined, recursively, as follows: any district is a region; if r_1 and r_2 are regions, then the union, intersection, or difference, of r_1 and r_2 are regions provided these are non-empty. A *zone* (or *minimal region*) is a region having no other region contained within it. Contours and regions denote sets.

A *spider* is a tree with nodes (called *feet*) placed in different zones; the connecting edges (called *legs*) are straight lines. A spider *touches* a zone if one of its feet appears in that region. A spider may touch a zone at most once. A spider is said to *inhabit* the region which is the union of the zones it touches. For any spider s, the *habitat* of s, denoted $\eta(s)$, is the region inhabited by s. The set of complete spiders within region r is denoted by $S(r)$. The set of spiders touching region r is denoted by $T(r)$. A spider denotes the existence of an element in the set denoted by the habitat of the spider. Two distinct spiders denote distinct elements.

Every region is a union of zones. A region is *shaded* if each of its component zones is shaded. A shaded region denotes the empty set if it is not touched by any spider. A *unitary SD2 diagram* is a single boundary rectangle together with a finite collection of contours (all possible intersections of contours must occur, i.e., the underlying diagram is a Venn diagram), spiders and shaded regions. Each contour must be labelled and no two contours in the same unitary diagram can have the same label. The labelling of spiders is optional. For any unitary diagram D, we use $C = C(D)$, $Z = Z(D)$, $Z^* = Z^*(D)$, $R = R(D)$, $R^* = R^*(D)$, $L = L(D)$ and $S = S(D)$ to denote the sets of contours, zones, shaded zones, regions, shaded regions, contour labels and spiders of D, respectively.

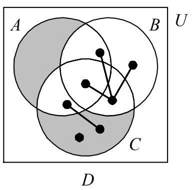

Figure 1

The SD2 diagram D in Figure 1 can be interpreted as:

$$A - (B \cup C) = \varnothing \wedge 1 \leq | C - (A \cup B) | \leq 2 \wedge 2 \leq | C - B | \wedge 1 \leq | B | .$$

2.2. Semantics of unitary SD2 diagrams

A *model* for a unitary SD2 diagram D is a pair $m = (\mathbf{U}, \Psi)$ where $\mathbf{U}$ is a set and $\Psi : C \to Set\ \mathbf{U}$, where $Set\ \mathbf{U}$ denotes the power set of $\mathbf{U}$, is a function mapping contours to subsets of $\mathbf{U}$. The boundary rectangle U is interpreted as $\mathbf{U}$.

A zone is uniquely defined by the contours containing it and the contours not containing it; its interpretation is the intersection of the sets denoted by the contours containing it and the complements of the sets denoted by those contours not containing it. We extend the domain of Ψ to interpret regions as subsets of $\mathbf{U}$. First define $\Psi : Z \to Set\ \mathbf{U}$ by

$$\Psi(z) = \bigcap_{c \in C^+(z)} \Psi(c) \cap \bigcap_{c \in C^-(z)} \overline{\Psi(c)}$$

where $C^+(z)$ is the set of contours containing the zone z, $C^-(z)$ is the set of contours not containing z and $\overline{\Psi(c)} = \mathbf{U} - \Psi(c)$, the *complement* of $\Psi(c)$. Since any region is a union of zones, we may define $\Psi : R \to Set\ \mathbf{U}$ by

$$\Psi(r) = \bigcup_{z \in Z(r)} \Psi(z)$$

where, for any region r, $Z(r)$ is the set of zones contained in r.

The semantics predicate $P_D(m)$ of a unitary diagram D is the conjunction of the following two conditions:

Distinct Spiders Condition: The cardinality of the set denoted by region r of unitary diagram D is greater than or equal to the number of complete spiders in r:

$$\bigwedge_{r \in R} |\Psi(r)| \geq |S(r)|$$

Shading Condition: The cardinality of the set denoted by a shaded region r of unitary diagram D is less than or equal to the number of spiders touching r:

$$\bigwedge_{r \in R^*} |\Psi(r)| \leq |T(r)|$$

2.3. Compound diagrams and multi-diagrams

Given two unitary diagrams D_1 and D_2, we can *connect* D_1 and D_2 with a straight line to produce a diagram $D = D_1 - D_2$. If a diagram has more than one rectangle, then it is a *compound* diagram. The 'connection operation' is commutative, $D_1 – D_2 = D_2 – D_1$. Hence, if a diagram has n unitary components, then these components can be placed in any order.

The semantics predicate of a compound diagram D is the disjunction of the semantics predicates of its component unitary diagrams; the boundary rectangles of the component unitary diagrams are interpreted as the same set **U**. That is,

$$P_D(m) = \bigvee_{i=1}^{n} P_{D_i}(m)$$

where $D = D_1 – D_2 – \ldots – D_n$.

Contours with the same labels in different unitary components of a compound diagram D are interpreted as the same set:

$$\forall c_1, c_2 \in C(D) \bullet \lambda(c_1) = \lambda(c_2) \Rightarrow \Psi(c_1) = \Psi(c_2)$$

where $\lambda(c)$ is the label of contour c.

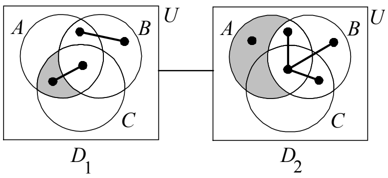

Figure 2

The compound diagram D in Figure 2 asserts that:

$$(\exists x, y \bullet x \in A \cap C \wedge y \in B - C \wedge |A \cap C - B| \le 1)$$
$$\vee (\exists x, y \bullet x \in B \wedge y \in A - (B \cup C)) \wedge |A - B| = 1.$$

A spider multi-diagram is a finite collection Δ of spider diagrams. The semantics predicate of a multi-diagram is the conjunction of the semantics predicates of the individual diagrams; the boundary rectangles of all diagrams are interpreted as the same set **U**. That is,

$$P_\Delta(\Psi) = \bigwedge_{D \in \Delta} P_D(\Psi).$$

Contours with the same labels in different individual diagrams of a multi-diagram Δ are interpreted as the same set:

$$\forall c_1, c_2 \in C(\Delta) \bullet \lambda(c_1) = \lambda(c_2) \Rightarrow \Psi(c_1) = \Psi(c_2).$$

2.4. Comparing regions across diagrams

Let D be a unitary SD2 diagram. For any $z \in Z(D)$, define $L^+(z) = \lambda(C^+(z))$, the set of labels of the contours containing z, and $L^-(z) = \lambda(C^-(z))$.

Given two unitary diagrams D and D', we can define equivalent regions across the two diagrams by considering partitions of the set of contour labels the two diagrams have in common. Let $P = (L^+, L^-)$ be a partition of $L(D) \cap L(D')$ and define $Z_P(D) = \{z \in Z(D) \mid L^+ = L^+(z) \cap L(D') \wedge L^- = L^-(z) \cap L(D')\}$. A region $zr \in R(D)$ is said to be *zonal with respect to* D' if there exists a partition P of $L(D) \cap L(D')$ such that

$$zr = \bigcup_{z \in Z_P(D)} z.$$

Suppose region zr of D is zonal with respect to D' and zr' of D' is zonal with respect to D. Then zr and zr' are *corresponding zonal regions*, denoted $zr \equiv_c zr'$, if there exists a partition P of $L(D) \cap L(D')$ such that

$$zr = \bigcup_{z \in Z_P(D)} z \quad \text{and} \quad zr' = \bigcup_{z' \in Z_P(D')} z'.$$

Let r be a region of D and let r' be a region of D'. Then r and r' are *corresponding regions*, denoted by $r' \equiv_c r$, if and only if r is a union of a set $ZR(r)$ of zonal regions with respect to D', r' is a union of a set $ZR(r')$ of zonal regions with respect to D, and

$$\forall zr \in ZR(r)\, \exists zr' \in ZR(r') \bullet$$
$$zr \equiv_c zr' \wedge \forall zr' \in ZR(r')\, \exists zr \in ZR(r) \bullet zr \equiv_c zr'$$

If $r_1 \in R(D)$ and $r_1 \subseteq r \equiv_c r'$, then r_1 is a *corresponding subregion* of r', denoted by $r_1 \subseteq_c r'$.

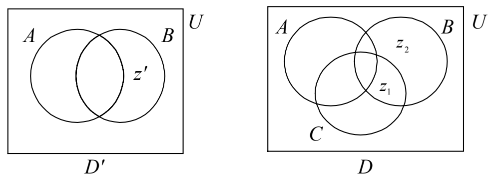

Figure 3

In Figure 3, the region $z = z_1 \cup z_2$ in D is zonal with respect to D' and the region z' in D' is zonal with respect to D. Furthermore, $z' \equiv_c z$ as both regions are associated with the partition $P = (\{B\}, \{A\})$ of $L(D) \cap L(D') = \{A, B\}$; hence $z_1 \subseteq_c z'$ and $z_2 \subseteq_c z'$.

Theorem 1 Corresponding regions are interpreted as the same set.

(i) $\forall r \in R(D)\, \forall r' \in R(D')\, \forall m = (\mathbf{U}, \Psi) \bullet$
$P_D(m) \wedge P_{D'}(m) \bullet r \equiv_c r' \Rightarrow \Psi(r) = \Psi(r')$.

(ii) $\forall r \in R(D)\, \forall r' \in R(D')\, \forall m = (\mathbf{U}, \Psi) \bullet$
$P_D(m) \wedge P_{D'}(m) \bullet r \subseteq_c r' \Rightarrow \Psi(r) \subseteq \Psi(r')$.

The proof is omitted. We can now give a definition of equivalent diagrams. Two unitary diagrams D and D' are *equivalent*, denoted by $D \equiv D'$, if

(i) $L(D) = L(D')$,

(ii) $\forall r \in R^*(D)\, \exists r' \in R^*(D') \bullet r \equiv_c r' \wedge$
$\forall r' \in R^*(D')\, \exists r \in R^*(D) \bullet r \equiv_c r'$ and

(iii) $\forall r \in R(D)\, \forall r' \in R(D') \bullet r \equiv_c r' \Rightarrow |S(r)| = |S(r')|$.

2.5. Compliance and Consistency

A model $m = (\mathbf{U}, \Psi)$ *complies* with diagram D if it satisfies its semantic predicate $P_D(m)$. We write $m \vDash D$. That is, $m \vDash D \iff P_D(m)$. Similarly, a model m *complies* with multi-diagram Δ if it satisfies its semantic predicate $P_\Delta(m)$. That is, $m \vDash \Delta \iff P_\Delta(m)$.

A diagram is *consistent* iff it has a compliant model. Similarly, a multi-diagram is consistent iff it has a compliant model.

Theorem 2 All SD2 diagrams are consistent.

The proof is based on the construction of topological models for the diagram. The details are omitted. Theorem 2 does not extend to multi-diagrams.

3. Diagrammatic reasoning rules

We introduce purely syntactic, diagrammatic rules for turning one diagram into another. In this section we define and illustrate the rules and show that they are valid.

3.1. Rules of transformation of unitary diagrams

We introduce rules that allow us to obtain one unitary diagram from a given unitary diagram by removing, adding or modifying diagrammatic elements.

Rule 1: Erasure of shading. We may erase the shading in an entire zone.

Rule 2: Erasure of a spider. We may erase a complete spider on any non-shaded region.

Figure 4 shows how the removal of a spider from a shaded region may result in an invalid inference. In diagram D, the set corresponding to region $A \cap B$ contains at most a single element, whereas in D', the corresponding set is empty.

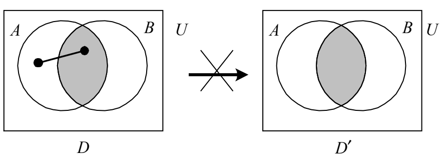

Figure 4

Rule 3: Erasure of a contour. We may erase a contour. When a contour is erased:

- any shading remaining in only a part of a zone should also be erased.
- if a spider has feet in two regions which combine to form a single zone with the erasure of the contour, then these feet are replaced with a single foot connected to the rest of the spider.

Rule 3 is illustrated in Figure 5.

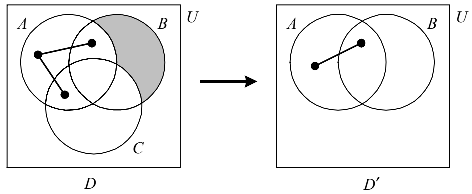

Figure 5

Rule 4: Spreading the feet of a spider. If a diagram has a spider s, then we may draw a node in any non-shaded zone z that does not contain a foot of s and connect it to s.

Rule 5: Introduction of a contour. A new contour may be drawn interior to the bounding rectangle observing the partial-overlapping rule: each zone splits into two zones with the introduction of the new contour. Each foot of a spider is replaced with a connected pair of feet, one in each new zone. Shaded zones become corresponding shaded regions.

3.2. Rules of transformation involving compound diagrams

Rule 6: Splitting spiders. If a unitary diagram D has a spider s whose habitat is formed by n zones, then we may replace D with a connection of n unitary diagrams D_1–…– D_n where each foot of the spider s touches a different corresponding zone in each diagram D_i.

Rule 6 is illustrated in Figure 6.

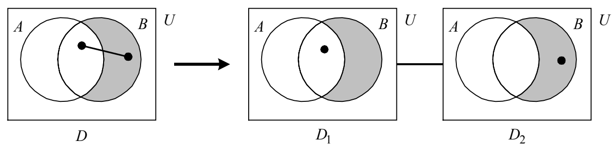

Figure 6

Rule 7: Rule of excluded middle. If a unitary diagram D has a non-shaded zone z where $|S(z)| = n$, then we may replace D with D_1–D_2, where D_1 and D_2 are unitary and one of the corresponding zones of z is shaded with $|S(z)| = n$ and the other is not shaded with $|S(z)| = n + 1$.

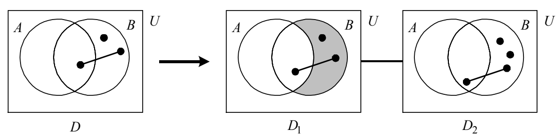

Figure 7

Rule 7 is illustrated in Figure 7. In diagram D, the set corresponding to region $B - A$ contains at least one element. In D_1 the set corresponding to $B - A$ contains either one or two elements and in D_2 it contains at least two elements.

Rule 8: The rule of connecting a diagram. For a given diagram D, we may connect any diagram D' to D.

Rule 9: The rule of construction. Given a diagram D_1–…–D_n, we may transform it into D if each $D_1,\ldots, D_n$ may be transformed into D by a sequence of the first eight transformation rules.

3.3. Consistency of a multi-diagram and the rule of inconsistency

Definition An α diagram is a diagram in which no spider's legs appear; that is, the habitat of any spider is a zone.

Any SD2 diagram D can be transformed into an α diagram by repeated application of rule 6, splitting spiders.

Theorem 3 (i) Two unitary α diagrams D^1 and D^2 with $L(D^1) = L(D^2)$ are consistent iff

$$\forall z_1 \in Z(D^1)\, \forall z_2 \in Z(D^2) \bullet z_1 \equiv_c z_2 \;\Rightarrow$$
$$\neg((z_1 \in Z^*(D^1) \wedge z_2 \in Z^*(D^2) \wedge |S(z_1)| \neq |S(z_2)|) \vee$$
$$(z_1 \in Z^*(D^1) \wedge z_2 \notin Z^*(D^2) \wedge |S(z_1)| < |S(z_2)|))$$

(ii) Let D^1 and D^2 be unitary but not α diagrams. Introduce contours, if necessary, into D^1 and D^2 to obtain D^{1a} and D^{2b}, where $L(D^{1a}) = L(D^{2b}) = L(D^1) \cup L(D^2)$. Transform D^{1a} and D^{2b} into their α diagrams $D_1^{1a} - \cdots - D_n^{1a}$ and $D_1^{2b} - \cdots - D_m^{2b}$. Then D^1 and D^2 are consistent iff there exist unitary components D_i^{1a} of D^{1a} and D_j^{2b} of D^{2b} such that D_i^{1a} and D_j^{2b} are consistent.

(iii) Two diagrams D^1 and D^2 are consistent iff there exist unitary components D_i^1 of D^1 and D_j^2 of D^2 such that D_i^1 and D_j^2 are consistent.

Case (iii) is the general case and relies on case (ii), which in turn relies on case (i). The formal details of the proof are again omitted. Intuitively, the diagrammatic condition in (i) would prevent the case in which two corresponding zones denote two sets whose cardinalities are inconsistent; this is the only case in which a pair of unitary α diagrams can be inconsistent.

Figure 8 shows a multi-diagram which is inconsistent, but whose components are pairwise consistent. Discussion of the consistency of multi-diagrams in general is deferred until we consider combining diagrams.

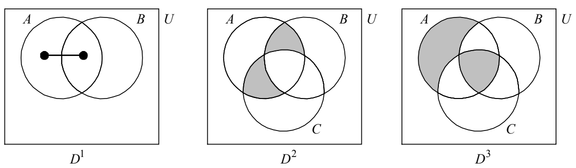

Figure 8

Rule 10: The rule of inconsistency. Given an inconsistent multi-diagram Δ, we may replace Δ with any multi-diagram.

3.4. Combining Diagrams

Given two consistent diagrams, D^1 and D^2, we can combine them to produce a diagram D, losing no semantic information in the process. In this section we describe the construction of such a combined diagram D. We give the rule for combining diagrams in several stages.

Rule 11: The rule of combining diagrams Let D^1 and D^2 be two consistent SD2 diagrams. Then their combination $D = D^1 * D^2$ is defined as follows.

(i) D^1 and D^2 are α unitary diagrams with $L(D^1) = L(D^2)$. The combined diagram D is also an α unitary diagram for which $L(D) = L(D^1) = L(D^2)$. So, for each $z \in Z(D)$, there exist corresponding zones $z_1 \in Z(D^1)$ and $z_2 \in Z(D^2)$. Furthermore, the number of spiders in z is equal to the maximum of the number of spiders in z_1 and the number of spiders in z_2, and z is shaded iff z_1 or z_2 is shaded.

$$\forall z \in Z(D)\, \exists z_1 \in Z(D^1)\, \exists z_2 \in Z(D^2) \bullet z \equiv_c z_1 \equiv_c z_2 \wedge$$
$$((z_1 \in Z(D^1) - Z^*(D^1) \vee z_2 \in Z(D^2) - Z^*(D^2) \;\Rightarrow$$
$$|S(z)| = \max(|S(z_1)|, |S(z_2)|) \wedge$$
$$(z_i \in Z^*(D^i) \Rightarrow |S(z)| = |S(z_i)|) \wedge$$
$$(z \in Z^*(D) \Leftrightarrow z_1 \in Z^*(D^1) \vee z_2 \in Z^*(D^2)))$$

(ii) D^1 and D^2 are unitary diagrams and $L(D^1) \neq L(D^2)$. We introduce contours into D^1 and D^2 to obtain D^{1a} and D^{2b}, where $L(D^{1a}) = L(D^{2b}) = L(D^1) \cup L(D^2)$. Transform D^{1a} and D^{2b} into their α diagrams $D_1^{1a} - \cdots - D_n^{1a}$ and $D_1^{2b} - \cdots - D_m^{2b}$. The combined diagram D is the compound diagram formed by combining each D_i^{1a} with each D_j^{2b}; where two components are inconsistent, we do not obtain a corresponding component in D.

(iii) D^1 and D^2 are any diagrams. The combined diagram D is the compound diagram formed by combining each component D_i^1 of D^1 with each component D_j^2 of D^2.

Again (iii) is the general case, relying on (ii) which in turn relies on (i). If D^1 and D^2 are inconsistent, then the combination is not defined. Rule 11(ii) is illustrated in Figure 9. First, contour C is added to D^2 to form D^{2a}, which is then transformed into an α diagram. We then combine the components of each diagram. Note that D^{1a} and D_2^{2a} are inconsistent, as are D^{1a} and D_3^{2a}, so the resulting combined diagram is unitary.

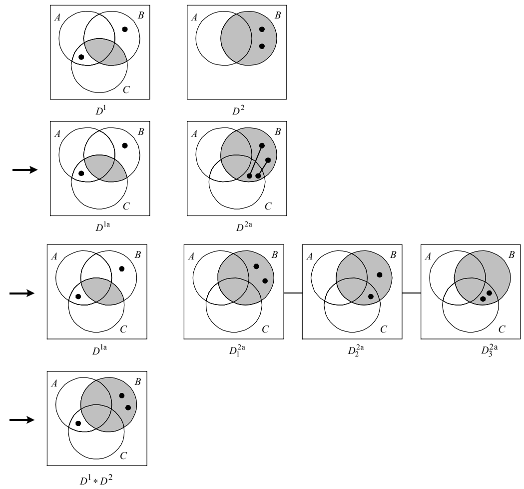

Figure 9

The associativity of * allows us to define the combination of the components of a multi-diagram $\Delta = \{D^1, D^2, \ldots, D^n\}$ unambiguously as D^* $= D^1 * D^2 * \cdots * D^n$. If Δ is inconsistent, the result will be no diagram; D^* is only defined when Δ is consistent. A test for the consistency of Δ is to try to evaluate D^*.

3.5. Soundness

D' is a consequence of D, denoted by $D \vDash D'$, if every compliant model for D is also a compliant model for D'. A rule is *valid* if, whenever a diagram D' is obtained from a diagram D by a single application of the rule then $D \vDash D'$. We write $\Delta \vdash D'$ to denote that diagram D' is obtained from multi-diagram Δ by a finite sequence of applications of the rules. We write $D \vdash D'$ to mean $\{D\} \vdash D'$, etc.

For space reasons, we omit the proofs of the validity of rules 1 to 11. These rules are similar to those of the Venn-II system given in [12] and SD1 [4] and the proofs are fairly straightforward. It can be noted that rules 5, introduction of a contour, 6, splitting spiders, 7, excluded middle, and 11, combining diagrams do not lose any semantic information; this is useful for proving completeness which we consider in the next section.

Theorem 4 Soundness Theorem Let Δ be a multi-diagram and D' a diagram. Then $\Delta \vdash D' \Rightarrow \Delta \vDash D'$.

The result follows by induction from the validity of the rules.

4. Completeness

To prove completeness we show that if diagram D' is a consequence of multi-diagram Δ, then Δ can be transformed into D' by a finite sequence of applications of the rules given in section 3. That is, $\Delta \vDash D' \Rightarrow \Delta \vdash D'$.

Definition A β diagram is an α diagram in which each zone is either shaded or contains at least one spider.

Any diagram D can be transformed into its β diagram D^{β} by repeated application of rule 6, splitting spiders, to turn it into an α diagram and then repeated application of rule 7, excluded middle, on empty zones, that is, on zones not containing spiders or shading. Figure 10 illustrates the transformation $D \vdash D^{\beta}$ for a unitary diagram D. Each zone in D^{β} conveys information; it tell us whether or not the corresponding set is empty.

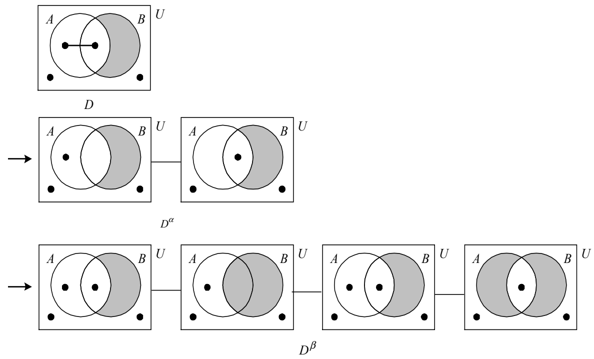

Figure 10

Theorem 5 For any diagram D, $D^{\beta} \vdash D$.

In SD1 the basic strategy was to expand D^*, the result of combining the diagrams of set Δ, and D' into β diagrams $D^{*\beta}$ and D'^{β} respectively. We can then show that for any unitary component $D_i^{*\beta}$ in $D^{*\beta}$ there is a unitary component $D_j'^{\beta}$ in D'^{β} which is a logical consequence of $D_i^{*\beta}$.

However, in SD2 the transformation of D^* and D' into β diagrams may not provide, for each unitary component $D_i^{*\beta}$ in $D^{*\beta}$, a unitary component $D_j'^{\beta}$ in D'^{β} such that $D_i^{*\beta} \vDash D_j'^{\beta}$. This is illustrated in Figure 11, where $D \vDash D_1'–D_2'$ but we can infer neither unitary component from D.

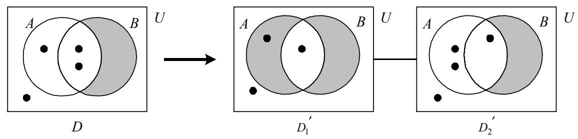

Figure 11

This difficulty arises in SD2 because rule 7, excluded middle, allows us to carry out the process of expanding a unitary component of a β diagram indefinitely provided it contains a non-shaded region. (In SD2 the only diagrams that cannot be split without adding contours are those where each region is shaded.)

To overcome this difficulty we apply rule 7 to the premise D to obtain D_1–D_2 such that $D_1 \vDash D_1'$ and $D_2 \vDash D_2'$. This is illustrated in Figure 12.

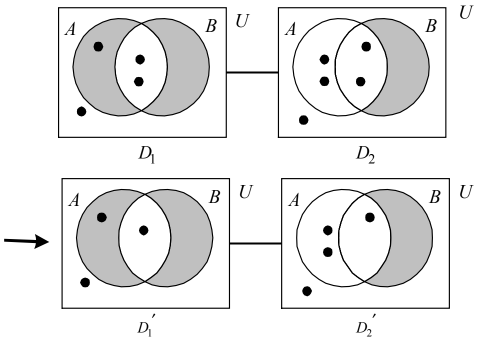

Figure 12

When we can show that for any unitary component D_i in D there is unitary component D_j in D'' which is a logical consequence of D_i, we can use Theorem 8, below, to show that $D_i \vdash D_j$.

First we state, without proof, the following theorems:

Theorem 6 Let $\Delta = \{D^1, D^2, \ldots, D^n\}$ and $D^* = D^1 * D^2 * \cdots * D^n$. Then $D^* \vDash \Delta$.

Theorem 7 Let D^c be the diagram obtained from D by rule 5, introduction of a contour. Then $D^c \vdash D$.

Theorem 8 Let D and D' be β unitary diagrams for which $L(D') = L(D)$. Then the following three statements are equivalent.

(i) $D \vdash D'$

(ii) $D \vDash D'$

(iii) [1] $\forall z' \in Z^*(D')\, \exists z \in Z^*(D) \bullet z \equiv_c z' \wedge |S(z)| = |S(z')|$

and [2] $\forall z' \in Z(D')\, \forall z \in Z(D) \bullet z' \equiv_c z \mid \Rightarrow |S(z)| \geq |S(z')|$.

Theorem 9 Let D and D' be compound diagrams for which each component is a β unitary diagram. That is,

$$D = D_1^{\beta} - D_2^{\beta} - \cdots - D_k^{\beta} \text{ and}$$

$$D' = D_1'^{\beta} - D_2'^{\beta} - \cdots - D_n'^{\beta}.$$

Assume further that

$\forall D_i^{\beta}\ \forall D_j'^{\beta} \bullet L(D_j'^{\beta}) = L(D_i^{\beta})$ and

$\forall z' \in Z^*(D_j'^{\beta}) \forall z \in Z(D_i^{\beta}) - Z^*(D_i^{\beta}) \bullet$

$z' \equiv_c z \Rightarrow |S(z')| < |S(z)|$.

Then $D \vDash D' \Rightarrow \forall D_i^{\beta}\ \exists D_j'^{\beta} \bullet D_i^{\beta} \vDash D_j'^{\beta}$.

Before proving the completeness theorem we outline the process of finding a derivation of diagram D' from a multidiagram Δ whenever D' is a logical consequence of Δ. First we replace Δ by D^*, the result of combining the components of multidiagram Δ. Secondly, we introduce contours in the premise, D^*, and the conclusion diagram, D', to obtain equivalent diagrams D^{*c} and D'^c, respectively, so that the sets of labels of any pair of unitary components in these diagrams are the same. After transforming D^{*c} and D'^c into their β diagrams, $D^{*c\beta}$ and $D'^{c\beta}$, we show that for any unitary component in the premise diagram, the conclusion follows. For each unitary component $D_i^{*c\beta}$ in the premise, we apply again rule 7, excluded middle, but this time to ensure that no shaded zone in the conclusion $D'^{c\beta}$ has a corresponding non-shaded zone in the premise containing an equal number of spiders.

By doing so we avoid the problem illustrated above in Figure 11. Now, there are diagrammatic conditions to show that there exists a unitary component in the conclusion deducible from the premise unitary component. And, further, by the rules of construction and connecting a diagram, and transitivity we can reverse the process and show that $D'^{c\beta}$ is derivable from $D^{*c\beta}$ and finally that D' is derivable from Δ.

Theorem 10 Completeness Theorem Let Δ be a multi-diagram and let D' be a diagram. Then $\Delta \vDash D' \Rightarrow \Delta \vdash D'$.

Proof If Δ is inconsistent, then the result follows immediately by applying Rule 10. Assume that Δ is consistent and that $\Delta \vDash D'$. By Theorem 6, $D^* \vDash \Delta$. So, by transitivity, $D^* \vDash D'$.

Introduce contours into each unitary component of D^* and D' to produce $D^{*c} = D_1^{*c} - D_2^{*c} - \cdots - D_m^{*c}$ and $D'^c = D_1'^c - D_2'^c - \cdots - D_n'^c$ so that $\forall D_i^{*c} \forall D_j'^c \bullet L(D_i^{*c}) = L(D_j'^c)$. Then, by Theorem 7 and soundness, $D^{*c} \vDash D^*$. By soundness $D' \vDash D'^c$. So, by transitivity, $D^{*c} \vDash D'^c$. Transform D^{*c} and D'^c into their β diagrams, $D^{*c\beta}$ and $D'^{c\beta}$, respectively. By Theorem 5 and the soundness theorem, $D^{*c\beta} \vDash D^{*c}$ and, by the soundness theorem, $D'^c \vDash D'^{c\beta}$. So, by transitivity, $D^{*c\beta} \vDash D'^{c\beta}$. Since $\forall D_i^{*c\beta}\ \forall D_j'^{c\beta} \bullet L(D_j'^{c\beta}) = L(D_i^{*c\beta})$, it follows from logical manipulation that $\forall D_i^{*c\beta} \bullet D_i^{*c\beta} \vDash D'^{c\beta}$. Apply the rule of excluded middle repeatedly to $D_i^{*c\beta}$ to obtain $D_{i1}^{*c\beta} - \ldots - D_{ip}^{*c\beta}$ so that

$$\forall z' \in Z^*(D_h'^{c\beta}) \forall z \in Z(D_{ik}^{*c\beta}) - Z^*(D_{ik}^{*c\beta}) \bullet$$

$$z' \equiv_c z \Rightarrow |S(z')| < |S(z)|.$$

It follows from Theorem 9 that $\forall D_{ik}^{*c\beta} \exists D_h'^{c\beta} \bullet D_{ik}^{*c\beta} \vDash D_h'^{c\beta}$. Hence, by Theorem 8, $\forall D_{ik}^{*c\beta} \exists D_h'^{c\beta} \bullet D_{ik}^{*c\beta} \vdash D_h'^{c\beta}$. So, by applying Rule 8, $\forall D_{ik}^{*c\beta} \bullet D_{ik}^{*c\beta} \vdash D'^{c\beta}$ and hence, by applying Rule 9, $D^{*c\beta} \vdash D'^{c\beta}$. Now, $\Delta \vdash D^*$, $D^* \vdash D^{*c\beta}$, and by Theorem 7, $D'^{c\beta} \vdash D'^{\beta}$, and by Theorem 5, $D'^{\beta} \vdash D'$, so, by transitivity, $\Delta \vdash D'$.

5. Conclusion and related work

We have given formal syntax and semantics and diagrammatic inference rules to the system of spider diagrams we call SD2. We have shown that the inference rules are sound and complete. In proving completeness, we have provided a proof strategy that should extend to most spider/constraint diagram systems and other similar systems based on Venn or Euler diagrams. Indeed, the proof can be adapted to give a simpler proof of the completeness of the Venn-II system than the one given by Shin.

We are in the process of proving soundness and completeness of other spider diagram systems using the strategy introduced in this paper. Our longer term aim is to prove similar results for constraint diagrams, and to provide the necessary mathematical underpinning for the development of software tools to aid the reasoning process.

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