Compiling Crosswords by SAT Solving

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Summer Project for MSc in Computer Science

[1]We thank Daniel Le Berre from the Université d'Artois for his help with sat4j

Outline of this talk

The crossword compiling problem

Davis-Putnam-Logemann-Loveland algorithm

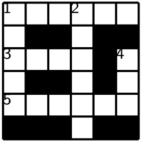
Encoding a crossword as a propositional formula

Refinements

The crossword compiling problem (fill-in crossword)

- Given a dictionary of words and a crossword grid:

torque	colon	tempt	bon	mini	pique
quirky	quay	any	encore	turkey	rue
clique	droopy	crypt	anyhow	yogi	would
loci	wreath	napkin	ugly		



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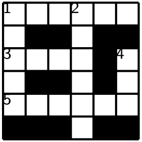

  find all the ways (if any) of arranging the words into the grid.
- Can the problem of tackled with a modern SAT solver, ie, can the problem be encoded such that:
  - the size of the formulae (the number of clauses) is not $O(\ldots d^2 \ldots)$, or worse, where d is the number of words in the dictionary?
  - the number of variables does not typically exceed 1000?

Davis-Putnam-Logemann-Loveland[2] (DPLL) algorithm

- Given a propositional formula, f say, does there exist a variable assignment (a model) under which f evaluates to true?
- Although SAT is NP-complete, efficient solvers do exist for many SAT instances [Stålmarck,US Patent N527689,1995]
- A model for $f = (\neg u \vee v) \wedge (\neg w \vee u) \wedge (\neg w \vee \neg v)$ is $\theta = \{u \mapsto \mathit{false}, v \mapsto \mathit{false}, w \mapsto \mathit{false}\}$

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- bool function DPLL(f, θ)

```
begin
    ⟨θ′, unsat⟩ = unit(f, θ);
    if (unsat) return false;
    else if (isSatisfied(f, θ′)) return true;
    else
        let x ∈ var(f) − var(θ′);
        if (DPLL(f, θ′ ∪ {x ↦ true})) return true;
        else return DPLL(f, θ′ ∪ {x ↦ false});
end
```

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Walk-through for $f = (\neg u \vee v) \wedge (\neg w \vee u) \wedge (\neg w \vee \neg v)$

Consider DPLL(f, θ_1) where $\theta_1 = \emptyset$

1. unit$(f, \theta_1) = \langle \theta_1', \mathit{false} \rangle$ where $\theta_1' = \emptyset$
2. isSatisfied$(f, \theta_1') = \mathit{false}$
3. Choose $w \in \mathit{var}(f) \setminus \mathit{var}(\theta_1') = \{u, v, w\} \setminus \emptyset = \{u, v, w\}$
4. Consider DPLL(f, θ_2) where $\theta_2 = \{w \mapsto \mathit{true}\}$
   - 4.1 unit$(f, \theta_2) = \langle \theta_2', \mathit{true} \rangle$ where $\theta_2' = \theta_2 \cup \{u \mapsto \mathit{true}, v \mapsto \mathit{false}\}$
   - 4.2 Thus DPLL$(f, \theta_2) = \mathit{false}$
5. Now consider DPLL(f, θ_2) where $\theta_2 = \{w \mapsto \mathit{false}\}$
   - 5.1 unit$(f, \theta_2) = \langle \theta_2', \mathit{false} \rangle$ where $\theta_2' = \{w \mapsto \mathit{false}\}$
   - 5.2 Choose $u \in \mathit{var}(f) \setminus \mathit{var}(\theta_2') = \{u, v, w\} \setminus \{w\} = \{u, v\}$
   - 5.3 Consider DPLL(f, θ_3) where $\theta_3 = \{w \mapsto \mathit{false}, u \mapsto \mathit{true}\}$
     - unit$(f, \theta_3) = \langle \theta_3', \mathit{false} \rangle$ and $\theta_3' = \theta_3 \cup \{v \mapsto \mathit{true}\}$
     - isSatisfied$(f, \theta_3') = \mathit{true}$
     - Thus DPLL$(f, \theta_3) = \mathit{true}$
   - 5.4 Thus DPLL$(f, \theta_2) = \mathit{true}$
6. Thus DPLL$(f, \theta_1) = \mathit{true}$

Some notes on the DPLL algorithm

- Solvers usually return the model and DPLL solvers can systematically enumerate all models;

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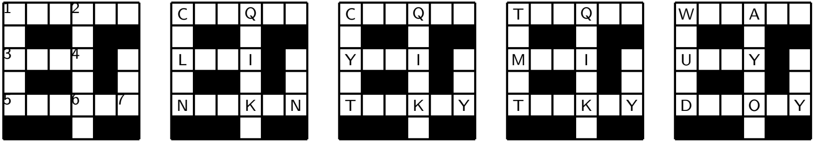
Some notes on the DPLL algorithm

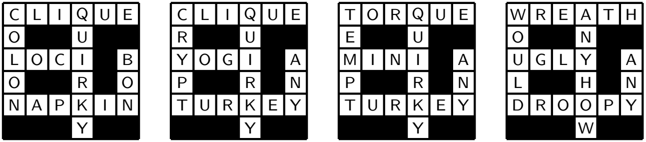
- Solvers usually return the model and DPLL solvers can systematically enumerate all models;
- More variables unit assigns, the more recursive calls are avoided;
- SAT is a "low-entry topic" because of the simplicity of DPLL;
- SAT research addresses topics such as:
  - Examining failing paths and adding new clauses to ensure that similar paths are not explored again;
  - Examining the structure of the SAT instance to assign variables in an intelligent order;
  - Investigating phase-transition behaviour;
  - SAT encoding and SAT applications

Encoding a crossword as a CNF formula (reduction)

- It is sufficient to find (encode) all combinations of characters that can arise at the intersection points between words



- Flesh out the words by searching the dictionary (note that two or more words might match the same intersection points)



Encoding a crossword as a CNF formula (compositionality)

- The 7 characters at intersection points can be represented by 35 propositional variables $x_1, \ldots, x_{35}$ where:
  - $\neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5$ expresses that intersection point 1 is character 0, ie, a
  - $\neg x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge x_9 \wedge \neg x_{10}$ expresses that intersection point 2 is character 2, ie, c
- Suppose that:
  - $f_1(x_1, \ldots, x_{10})$ expresses the relationships between points 1 and 2 imposed by the horizontal starting at square 1;
  - $f_2(x_1, \ldots, x_5, x_{11}, \ldots, x_{15}, x_{21}, \ldots, x_{25})$ between points 1, 3 and 5 imposed by the vertical starting at square 1;
  - ...
  - $f_6(x_{30}, \ldots, x_{35})$ expresses the relationships on point 7 imposed by the vertical ending at square 7;

[Draw grid with intersection points on board]

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  - ...
  - $f_6(x_{30}, \ldots, x_{35})$ expresses the relationships on point 7 imposed by the vertical ending at square 7;
- Then $f_1(x_1, \ldots, x_{35}) \wedge \ldots \wedge f_6(x_1, \ldots, x_{35})$ is a *CNF* formula that expresses the relationships between all intersection points

[Draw grid with intersection points on board]

Generating the formula $f_1(x_1, \ldots, x_{10})$

- Scan the dictionary for all 6 letter words and extract the first and fourth characters:

torque	colon	tempt	bon	mini	pique	tq	qr	eo
quirky	quay	any	*encore*	*turkey*	rue	tk	cq	do
clique	*droopy*	crypt	*anyhow*	yogi	would	ah	wa	nk
loci	wreath	*napkin*	ugly					

- Interpret as 10-bit numbers, sort and encode as a formula:

ah	00000,00111
cq	00010,10000
...	
qr	10000,10001
wa	10111,00000

$$f_1 = \bigvee \begin{cases} \neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \; x_8 \wedge \; x_9 \wedge \; x_{10} & 00000, 00111 \\ \neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \; x_4 \wedge \neg x_5 \wedge \; x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge \neg x_9 \wedge \neg x_{10} & 00010, 10000 \\ \quad \ldots & \\ x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \; x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge \neg x_9 \wedge \; x_{10} & 10000, 10001 \\ x_1 \wedge \neg x_2 \wedge \; x_3 \wedge \; x_4 \wedge \; x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge \neg x_9 \wedge \neg x_{10} & 10111, 00000 \end{cases}$$

Generating the formula $f_1(x_1, \dots, x_{10})$ (reprise)

Alternatively $\neg f_1 = g_0 \vee g_1 \vee \dots \vee g_9$ where g_i are in DNF and:

$$g_0 = \bigvee \begin{cases} \neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge \neg x_9 \wedge \neg x_{10} & 00000, 00000 \\ \neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge \neg x_9 \wedge x_{10} & 00000, 00001 \\ \neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge x_9 \wedge \neg x_{10} & 00000, 00010 \\ \neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge x_9 \wedge x_{10} & 00000, 00011 \\ \neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge x_8 \wedge x_9 \wedge x_{10} & 00000, 00100 \\ \neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge x_8 \wedge \neg x_9 \wedge x_{10} & 00000, 00101 \\ \neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge x_8 \wedge x_9 \wedge \neg x_{10} & 00000, 00110 \end{cases}$$

$$g_0 = \bigvee \begin{cases} \neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge \neg x_9 & 00000, 0000* \\ \neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge x_9 & 00000, 0001* \\ \neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge x_8 \wedge \neg x_9 & 00000, 0010* \\ \neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge x_8 \wedge x_9 \wedge \neg x_{10} & 00000, 00110 \end{cases}$$

where the second g_0 is compromised of 4 implicants.

Generating the formula $g_1(x_1, \ldots, x_{10})$

$$g_1 = \bigvee \begin{cases} \neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge \neg x_9 \wedge \neg x_{10} & 00000, 01000 \\ \neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge \neg x_9 \wedge \quad x_{10} & 00000, 01001 \\ \neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge \quad x_9 \wedge \neg x_{10} & 00000, 01010 \\ \quad \ldots & \\ \neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \quad x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \quad x_7 \wedge \quad x_8 \wedge \neg x_9 \wedge \quad x_{10} & 00010, 01101 \\ \neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \quad x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \quad x_7 \wedge \quad x_8 \wedge \quad x_9 \wedge \neg x_{10} & 00010, 01110 \\ \neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \quad x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \quad x_7 \wedge \quad x_8 \wedge \quad x_9 \wedge \quad x_{10} & 00010, 01111 \end{cases}$$

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where the first and second g_1 compromise of $1010000_2 - 111_2 - 1 = 72$ and 4 implicants respectively.

This way of obtaining a CNF encoding cannot be novel.

Complexity of the encoding

- The formulae g_0 and g_9 consists of $\leq 2 \times 5$ implicants and all other g_i consist of $\leq 2 \times 2 \times 5$ implicants
- More generally, each $\neg g_i$ consists of $O(\lg(c)m)$ clauses where:
  - c is the number of characters in the alphabet
  - m is the maximum number of intersections for any word in the grid
- Each f_i consists of $O(\lg(c)md)$ clauses and the complete system is $\wedge_i f_i$ is $O(\lg(c)mdg)$ where:
  - d is the number of words in the dictionary
  - g is the number of words in the grid

Dictionary of 73,338 words on a 60 word grid with 132 intersections

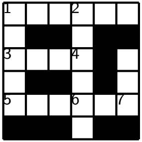
G		E		A	R	S	I	S	S	E		A	M	A	R	A	S
A	G	R	U	N		U		U		O		D		V		G	
U		E		H	A	B	E	B	A	S		F	A	U	O	R	E
D	U	P	L	E	X		A		D			O		L		U	
I		T		L	E	G	E		H	O	R	R	I	S	O	N	O
A	M	A	V	I		R			I		E		T	E	R		N
				T	R	A	N	A	B	O	Q	U	E		A	X	E
P	I	O		U		V			E		U		S	I	N		R
R		U	A	S	T	A	B	A	R		I				T	U	E
O	R	A				V		V	I	D	E	T	Q	U	E		N
P		N	I	L		I		E			S		U		S	I	T
A	S	T		A	B	S	O	L	U	O	Q	U	E				
G		E	A	T		S		L			U		R	U	E	R	E
O	B	S	T	A	V	E	R	E		S	E	S	E		X		V
	R		T		E			R		U		E	M	O	T	A	E
M	E	T	U	I	S		S	E	N	A	T	U	I		E		R
	V		L		T		P		E		U		N	O	R	I	S
R	E	G	I	A	E		E	R	U	D	I	R	I		A		A

- 1,092,868 clauses generated in 76s and sat4j solves the SAT instance in 367s $\approx$ 6m on a 3.2GHz, 1GB RAM PC
- French requires 6-bit encoding for á, â, ç, è, é, ê, ô, œ, etc

Reducing the number of propositional variables

Consider again the dictionary and the grid:

torque	colon	tempt	bon	mini	pique
quirky	quay	any	encore	turkey	rue
clique	droopy	crypt	anyhow	yogi	would
loci	wreath	napkin	ugly		



- $S_1 \subseteq \{A, C, D, E, N, Q, T, W\}$ and $S_2 \subseteq \{A, K, H, K, O, Q, R\}$

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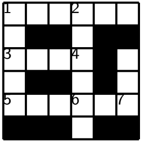

- $S_1 \subseteq \{A, C, D, E, N, Q, T, W\}$ and $S_2 \subseteq \{A, K, H, K, O, Q, R\}$
- $S_4 \subseteq \{I, Y\}$ and $S_6 \subseteq \{K, O\}$ from quirky and anyhow

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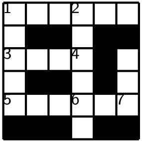

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- $S_5 \subseteq \{D, E, N, T\}$ and $S_7 \subseteq \{E, N, Y\}$ from encore, turkey, droopy and napkin

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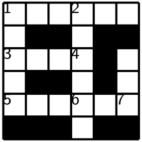

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- $S_4 \subseteq \{I, Y\}$ and $S_6 \subseteq \{K, O\}$ from quirky and anyhow
- $S_5 \subseteq \{D, E, N, T\}$ and $S_7 \subseteq \{E, N, Y\}$ from encore, turkey, droopy and napkin
- $S_3 \subseteq \{L, M, Q, U, Y\}$ from mini, quay, yogi, loci and ugly

Reducing the number of propositional variables

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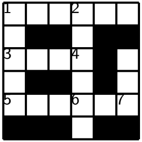

- $S_1 \subseteq \{A, C, D, E, N, Q, T, W\}$ and $S_2 \subseteq \{A, K, H, K, O, Q, R\}$
- $S_4 \subseteq \{I, Y\}$ and $S_6 \subseteq \{K, O\}$ from quirky and anyhow
- $S_5 \subseteq \{D, E, N, T\}$ and $S_7 \subseteq \{E, N, Y\}$ from encore, turkey, droopy and napkin
- $S_3 \subseteq \{L, M, Q, U, Y\}$ from mini, quay, yogi, loci and ugly
- $S_1 \subseteq \{C, T, W\}$ from colon, tempt, pique, crypt and would (note how the P is excluded)

Minimising the number of propositional variables

```
for i := 1 to 7    {    s[i] := {a, ..., z}    }
change := true
while change
    change := false
    for all w ∈ {1a, 1d, 2d, 3a, 5a, 4d}
        suppose w includes intersections i_1, ..., i_k at positions p_1, ..., p_k
        for j := 1 to k    {    t[j] = ∅    }
        read word d from dictionary until empty
            if length(d) = length(w) then
                keep := true
                for j := 1 to k
                    if char(d, p_j) ∉ s[i_j] then keep := false
                if keep then
                    for j := 1 to k    {    t[j] = t[j] ∪ {char(d, p_j)}    }
        for j := 1 to k
            if s[i_j] ∩ t[j] ⊂ s[i_j] then
                change := true; s[i_j] := s[i_j] ∩ t[j]
```

Avoiding the SAT encoding with divide-and-conquer

- Minimise S_i
- If there exists $S_i = \emptyset$ then return []
- If each $S_i = \{c_i\}$ then return $[[c_1, \ldots, c_7]]$
- Otherwise there exists $S_i = \{c_1, \ldots, c_k\}$ where $k > 1$ then
  - Put $S_i = \{c_1, \ldots, c_{\lceil k/2 \rceil}\}$ and recurse to obtain L_1
  - Put $S_i = \{c_{\lceil k/2 \rceil + 1}, \ldots, c_k\}$ and recurse to obtain L_2
- Return $\mathit{append}(L_1, L_2)$

[Relevance of principle of least commitment]

Time for a demonstration

java15 -Xmx300m -jar CrossWord.jar