Calculating Convex Hulls with a Linear Solver

With thanks to Mats Carlsson, Bart Demoen, Raimund Seidel and Axel Simon

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Calculating convex hulls

- The problem of calculating the convex hull of a set of points $\{x_1, \ldots, x_n\}$ is almost trivial
- Consider, however, the problem of calculating the smallest polyhedron P that includes $P_1 \cup P_2$ where P_1 and P_2 are represented in standard form:

$$P_1 = \{\vec{x} \in \mathbb{R}^n \mid A_1\vec{x} \leq \vec{B}_1\} \qquad P_2 = \{\vec{x} \in \mathbb{R}^n \mid A_2\vec{x} \leq \vec{B}_2\}$$

Recall that $P \subseteq \mathbb{R}^n$ is polyhedral iff it can be represented as the intersection of a finite number of half-spaces, that is, as $P = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} \leq \vec{B}\}$.

Convex hull of $P_1 \cup P_2$ is not closed

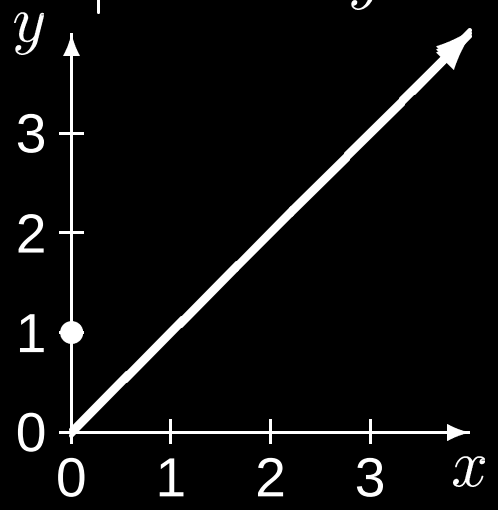
Consider P_1 and P_2 where

$$P_1 = \left\{ \vec{x} \in \mathbb{R}^2 \;\middle|\; \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \vec{x} \leq \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

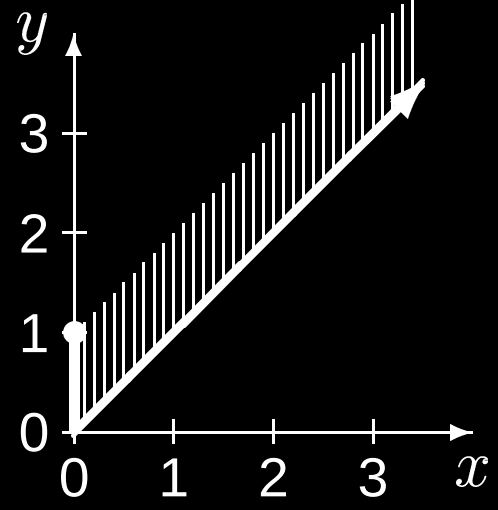
$$P_2 = \left\{ \vec{x} \in \mathbb{R}^2 \;\middle|\; \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -1 & 0 \end{bmatrix} \vec{x} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Convex hull of $P_1 \cup P_2$ is not closed

The convex hull of $P_1 \cup P_2$ excludes the points $\{\langle x, y\rangle \in \mathbb{R}^2 \mid x > 0 \wedge y = x + 1\}$ and hence is not closed.



P_1 and P_2



convex hull of $P_1 \cup P_2$

Since the convex hull of $P_1 \cup P_2$ is not necessarily closed, the convex hull cannot always be represented by a system of non-strict linear inequalities. Thus apply closure.

Classic approach to computing the closure of the hull

- The classic approach to computing the closure of the convex hull is to employ two representations: (i) points and rays and (ii) systems of non-strict inequalities;
- The Chernikova algorithm [Le Verge, 1992] can convert between (i) and (ii) in both directions;
- Thus convert from (ii) to (i), then calculate closure of the convex hull, then convert from (i) to (ii).

Computing the closure of the hull

The convex hull of $P_1 \cup P_2$ is given by:

$$P_H = \left\{ \vec{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \vec{x} = \sigma_1 \vec{x}_1 + \sigma_2 \vec{x}_2 \wedge \sigma_1 + \sigma_2 = 1 \wedge 0 \leq \sigma_1 \wedge \\ A_1 \vec{x}_1 \leq \vec{B}_1 \qquad\quad \wedge A_2 \vec{x}_2 \leq \vec{B}_2 \;\; \wedge 0 \leq \sigma_2 \end{array} \right\}$$

This system can be relaxed by putting $\vec{y}_1 = \sigma_1 \vec{x}_1$ and $\vec{y}_2 = \sigma_2 \vec{x}_2$ so that $\vec{x} = \vec{y}_1 + \vec{y}_2$ and $A_i \vec{y}_i \leq \sigma_i \vec{B}_i$ to define:

$$P_{CH} = \left\{ \vec{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \vec{x} = \vec{y}_1 + \vec{y}_2 \;\; \wedge \sigma_1 + \sigma_2 = 1 \;\; \wedge 0 \leq \sigma_1 \wedge \\ A_1 \vec{y}_1 \leq \sigma_1 \vec{B}_1 \wedge A_2 \vec{y}_2 \leq \sigma_2 \vec{B}_2 \wedge 0 \leq \sigma_2 \end{array} \right\}$$

Note that P_{CH} is polyhedral since it is the projection of a system of non-strict linear inequalities.

P_{CH} is the closure of the hull of $P_1 \cup P_2$

To see this, consider the recession cone of a polyhedron P, that is, $0^+P = \{\vec{y} \in \mathbb{R}^n \mid \forall \lambda \geq 0 \,.\, \forall \vec{x} \in P \,.\, \vec{x} + \lambda \vec{y} \in P\}$.

Theorem 19.6 of Rockafellar states that the closure of the convex hull of $P_1 \cup P_2$ is the set

$$(0^+P_1 + P_2) \cup (P_1 + 0^+P_2) \cup$$
$$(\cup\{\sigma_1 P_1 + \sigma_2 P_2 \mid \sigma_1 + \sigma_2 = 1 \wedge 0 < \sigma_1, \sigma_2\})$$

Intuitively, $0^+P_1 + P_2$ is P_2 extended in the directions of half-lines contained within P_1.

P_{CH} is the closure of the hull of $P_1 \cup P_2$

Let $\vec{x} \in P_i$. Now $\vec{y} \in 0^+P_i$ iff $A_i(\vec{x} + \lambda\vec{y}) \leq \vec{B}_i$ for all $\lambda \geq 0$ which holds iff $A_i\vec{y} \leq \vec{0}$.

Therefore

$0^+P_1 + P_2 = \{\vec{x} \in \mathbb{R}^n \,|\, \vec{x} = \vec{y}_1 + \vec{y}_2 \wedge A_1\vec{y}_1 \leq \vec{0} \wedge A_2\vec{y}_2 \leq \vec{B}_2\}$.

Observe that

$\{\vec{x} \in \mathbb{R}^n \mid \vec{x} = \vec{y}_1 + \vec{y}_2 \wedge A_1\vec{y}_1 \leq \sigma_1\vec{B}_1 \wedge A_2\vec{y}_2 \leq \sigma_2\vec{B}_2\}$

coincides with the sets (i) $0^+P_1 + P_2$, (ii) $P_1 + 0^+P_2$ and (iii) $\cup\{\sigma_1P_1 + \sigma_2P_2 \mid \sigma_1 + \sigma_2 = 1 \wedge 0 < \sigma_1, \sigma_2\}$ when (i) $\sigma_1 = 0$ and $\sigma_2 = 1$, (ii) $\sigma_1 = 1$ and $\sigma_2 = 0$ and (iii) $\sigma_1 + \sigma_2 = 1$ and $0 < \sigma_1, \sigma_2$ respectively.

Therefore P_{CH} is the closure of the convex hull.

Towards a linear solver-based implementation

This result leads to an algorithm for computing the closure of the convex hull:

- construct the systems $A_i\vec{y}_i \leq \sigma_i\vec{B}_i$ by scaling the constant vectors $\vec{B}_i$ by σ_i,
- add the constraints $\vec{x} = \vec{y}_1 + \vec{y}_2$, $\sigma_1 + \sigma_2 = 1$ and $0 \leq \sigma_i$,
- eliminate variables other than $\vec{x}$ using projection to obtain P_{CH} in terms of $\vec{x}$.

Hence the closure of the convex hull can be computed without recourse to another representation.

Non-ground representation

- Closed polyhedra will be represented by lists (conjunctions) of linear constraints.
- A linear constraint c and an expression e takes the form

$$c ::= e \leq e | e = e | e \geq e$$

$$e ::= x | n | n * x | -e | e + e | e - e$$

  where n is a rational number and x is a variable.
- Adding constraints to a system then amounts to list concatenation.

Projection

We need to construct a predicate `project(+Xs,+Cxs,-ProjectCxs)` that is true when for a given list of dimensions `Xs` and a given list of constraints `Cxs`, `ProjectCxs` is the projection of `Cxs` onto `Xs`.

Fortunately, the CLP($\mathbb{R}$) library provides a projection facility, `dump`, that can be used as in the query `{Y > X}, dump([X, Y], NewVars, Answer)` that binds `NewVars` to `[A, B]` and `Answer` to `[A < B]`.

Projection code

```
project(Xs, Cxs, ProjectCxs) :-
  tell_cs(Cxs),
  dump(Xs, Vs, ProjectCxs), Xs = Vs.

tell_cs([]).
tell_cs([C|Cs]) :- {C}, tell_cs(Cs).
```

For example, the query `project([X, Z], [X < Y, Y < Z], ProjectCs)` will bind `Cs` to `[X-Z<0]`.

However, the compound query `{X = Z + 1}, project([X, Z], [X < Y, Y < Z], ProjectCs)` will fail because of store interaction.

Projection code

To insulate the constraints posted in `tell_cs`, both the variables `Xs` and the constraints `Cxs` need to be renamed.

```
project(Xs, Cxs, ProjectCxs) :-
  copy_term(Xs-Cxs, CpyXs-CpyCxs),
  tell_cs(CpyCxs),
  dump(CpyXs, Vs, ProjectCxs), Xs = Vs.
```

Note that care must be taken to ensure that `Xs` and `Cxs` are renamed consistently.

Code for closure of hull code

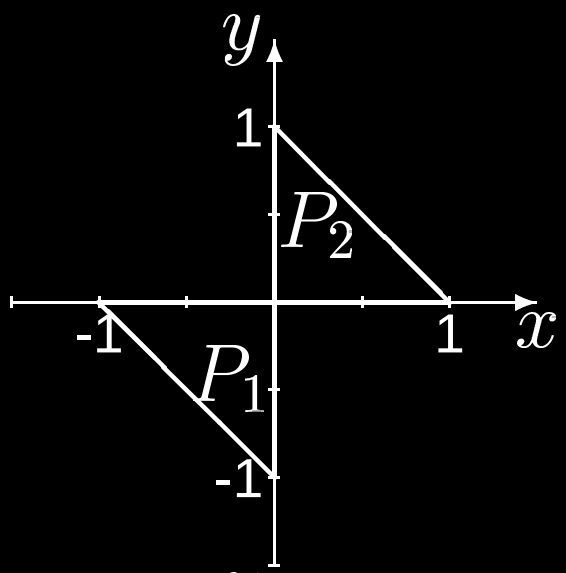
```
convex_hull(Xs, Cxs, Ys, Cys, Zs, Czs) :-
  scale(Cxs, Sig1, [], C1s),
  scale(Cys, Sig2, C1s, C2s),
  add_vect(Xs, Ys, Zs, C2s, C3s),
  project(Zs, [Sig1 >= 0, Sig2 >= 0, Sig1+Sig2 = 1|C3s],

add_vect([], [], [], Cs, Cs).
add_vect([U|Us], [V|Vs], [W|Ws], C1s, C2s) :-
    add_vect(Us, Vs, Ws, [W = U+V|C1s], C2s).
```

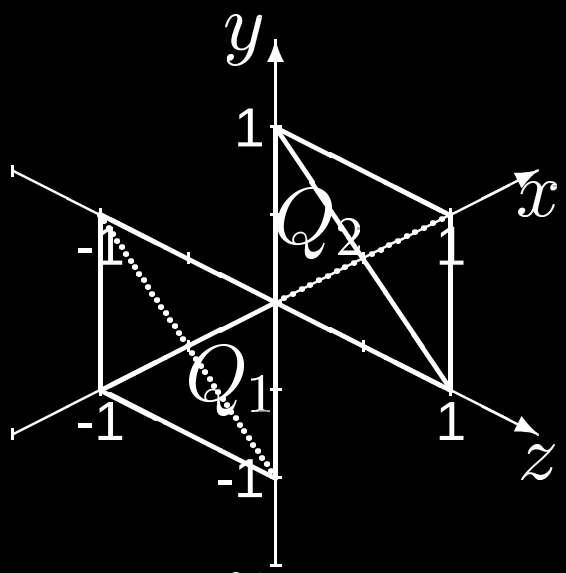
Running this code on the original example yields the query `convex_hull([X1,Y1],[X1=0,Y1=1],[X2,Y2], [X2>=0,Y2=X2],[A,B],S)` which gives the answer `S = [A>=0,A-B>=-1,A-B=<0]`.

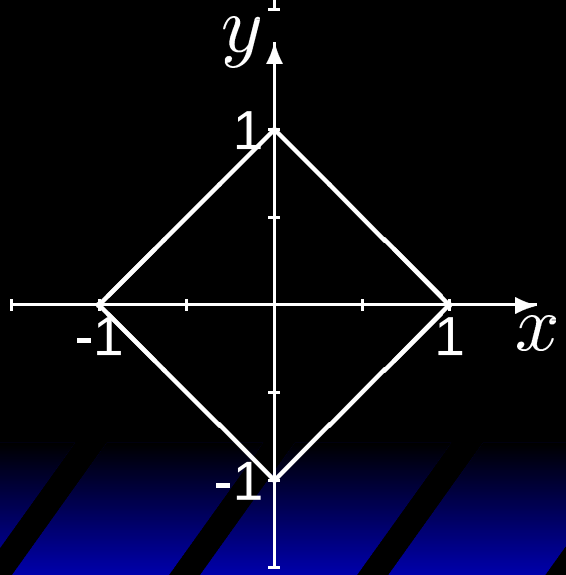
Complexity of closure of the hull

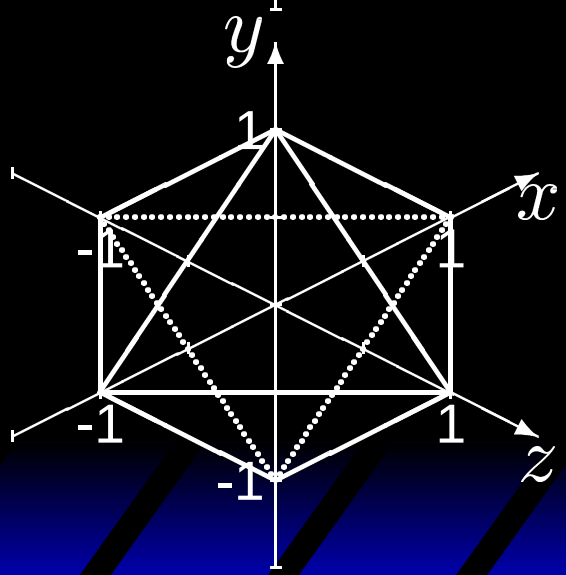
- The algorithm relies on project – which is implemented Fourier-Motzkin style – which is known to be exponential;
- It was recently shown [Chandru, 2000] that linear projection is fundamentally exponential;
- The problem of converting between points and rays and systems of non-strict inequalities is also intrinsically exponential.

Complexity of closure of the hull









Conclusions

- Despite the scaling problems, the technique has been widely applied in logic programming, mostly to satisfaction.
- In the context of termination inference, this method is feasible since it accounts for 42% of the first pass of the analysis and the first pass itself accounts for only 23% of the total analysis time.
- In principle the tactic is very simple since it exploits built in machinery; in fact it will appear as a pearl in TPLP early in 2005.