Coeffects: contextual effects / the dual of effects

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but also….

Aloıs Brunell, Marco Gaboardi,
Damiano Mazza, Steve Zdancewic

“A core quantitative coeffect calculus” (ESOP 2013)

Dan Ghica, Alex Smith

“Bounded linear types” (ESOP 2013)

Context in programming

and its (static) analysis

- Free variables (and their types)

  Simply-typed λ-calculus $\boxed{\Gamma} \vdash e : \tau$

  $\{x_1 : \tau_1, \ldots, x_n : \tau_n\}$

- Free variables (types and <u>usage</u>)

  Linear types $\Gamma \vdash e : \sigma \boxed{\multimap} \tau$

  Bounded linear types $\Gamma \vdash e : \boxed{!_{\mathrm{n}}}\sigma \to \tau$

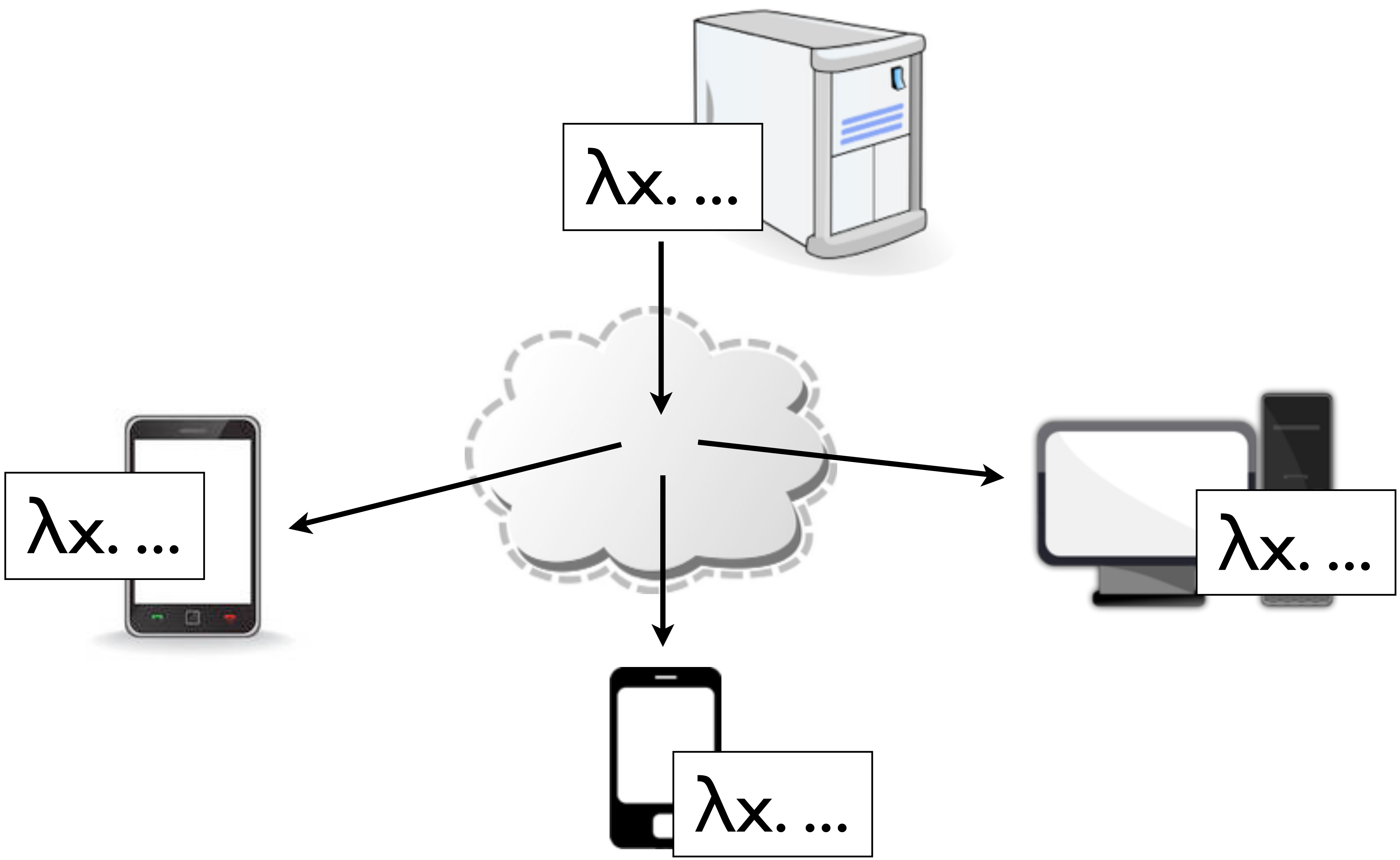
- Compilation context (e.g., overloading)

  Type class constraints $(+) :: \boxed{\mathtt{Num\ a}} \Rightarrow \mathtt{a} \to \mathtt{a} \to \mathtt{a}$

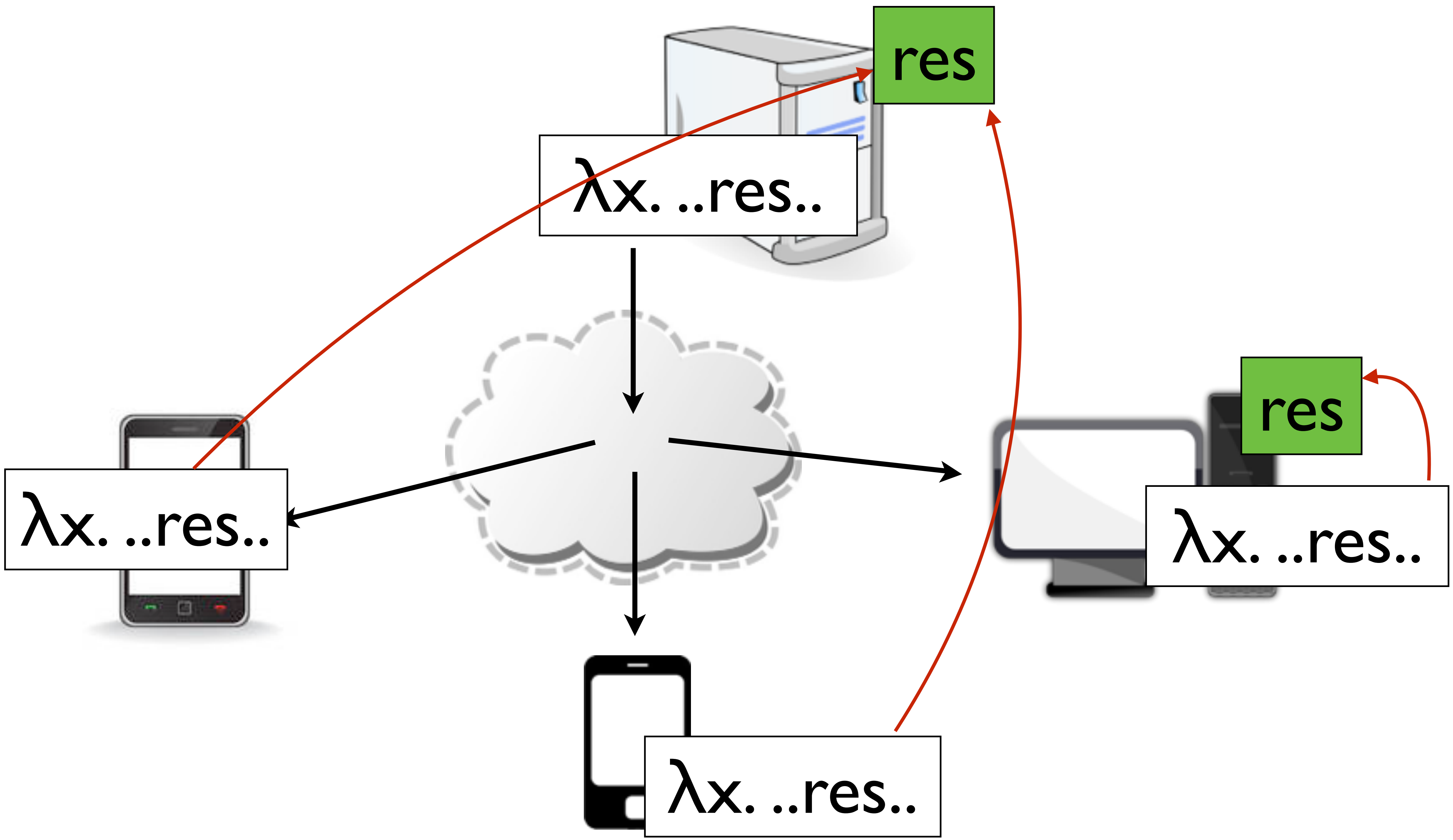
- Run-time context (e.g., resources)

  Input-output requirements, sensors, databases…

Context: distributed programming



Context: distributed programming



The *coeffect calculus*

- Simply-typed λ-calculus for contextual computations
  - analysis: type and *coeffect system*
  - analysis-directed semantics: via *indexed comonads* ++

(parametric coeffect comonads)

(graded comonads)

- Coeffect : d*ualises* effects (+ monads)

Monads + effects

$$\Gamma \vdash e : \mathbf{M}_{F}\tau$$

$$\text{abs}\ \frac{\Gamma, x : \sigma \vdash e : \mathbf{M}_{F}\tau}{\Gamma \vdash \lambda x\ .\ e : \mathbf{M}_{\emptyset}(\sigma \rightarrow \mathbf{M}_{F}\tau)} \qquad \text{var}\ \frac{x : \sigma \in \Gamma}{\Gamma \vdash x : \mathbf{M}_{\emptyset}\sigma}$$

lambda delays effects

$$\text{app}\ \frac{\Gamma \vdash e_1 : \mathbf{M}_{F}(\sigma \rightarrow \mathbf{M}_{G}\tau) \qquad \Gamma \vdash e_2 : \mathbf{M}_{H}\sigma}{\Gamma \vdash e_1\ e_2 : \mathbf{M}_{(F \sqcup G \sqcup H)}\tau}$$

[see Katsumata, POPL 2014]

Dualising

$$\mathrm{A} \leftarrow B \quad \Big\| \quad \mathrm{A} \rightarrow B$$

Productive/output effects $[\,\Gamma \vdash e : \tau, \mathsf{F}\,] : [\Gamma] \rightarrow \mathbf{M}_{\mathsf{F}}[\tau]$

structure on the right

Consuming/input effects $[\,\Gamma \mathrel{?} \mathsf{R} \vdash e : \tau\,] : \mathbf{C}_{\mathsf{R}}[\Gamma] \rightarrow [\tau]$

Coeffects

structure on the left

But… λ-calculus is asymmetric: many-to-one

Design question: structure over all Γ or per variable?

answer : yes! either

Coeffects: two varieties (shapes)

$$\Gamma \mathbin{?} \mathsf{R} \vdash e : \tau$$

per variable
local

shape = $|\Gamma|$

$\mathsf{R} = \langle r_1, \ldots, r_n \rangle$

$$\textit{where } \Gamma = \{x_1 : \tau_1, \ldots, x_n : \tau_n\}$$

$$\boxed{\mathsf{C}_{r1}[\tau_1] \times \ldots \times \mathsf{C}_{rn}[\tau_n] \to [\tau]}$$

per context
global

shape = 1

$\mathsf{R} = r_1$

$$\boxed{\mathsf{C}_{r1}[\Gamma] \to [\tau]}$$

Example: reuse bounds bounded linear logic

- Per variable
- Track number of uses of a variable
- Coeffects annotations are vectors of natural numbers

Example: reuse bounds bounded linear logic

abs
$$\frac{\Gamma,\ x : \sigma \mathrel{?} \mathsf{R} \times \langle \mathsf{s} \rangle \vdash e : \tau}{\Gamma \mathrel{?} \mathsf{R} \vdash \lambda x.e : \sigma \xrightarrow{\mathsf{s}} \tau}$$

var
$$\frac{}{x : \sigma \mathrel{?} \langle 1 \rangle \vdash x : \sigma}$$

app
$$\frac{\Gamma_1 \mathrel{?} \mathsf{R} \vdash e_1 : \sigma \xrightarrow{\mathsf{t}} \tau \qquad \Gamma_2 \mathrel{?} \mathsf{S} \vdash e_2 : \sigma}{\Gamma_1, \Gamma_2 \mathrel{?} \mathsf{R} \times (\mathsf{t} * \mathsf{S}) \vdash e_1\, e_2 : \tau}$$

contr
$$\frac{\Gamma_1,\ x : \sigma,\ y : \sigma,\ \Gamma_2 \mathrel{?} \mathsf{R} \times \langle \mathsf{a,b} \rangle \times \mathsf{S} \vdash e : \tau}{\Gamma_1,\ z : \sigma,\ \Gamma_2 \mathrel{?} \mathsf{R} \times \langle \mathsf{a+b} \rangle \times \mathsf{S} \vdash e[z \leftarrow x, z \leftarrow y] : \tau}$$

weak
$$\frac{\Gamma \mathrel{?} \mathsf{R} \vdash e : \tau}{\Gamma, x : \sigma \mathrel{?} \mathsf{R} \times \langle 0 \rangle \vdash e : \tau}$$

sub
$$\frac{\Gamma \mathrel{?} \mathsf{R} \vdash e : \tau \qquad \mathsf{R} \leq \mathsf{R'}}{\Gamma \mathrel{?} \mathsf{R'} \vdash e : \tau}$$

Example: reuse bounds

$$\text{const}\ \frac{}{\varnothing \mathrel{?} \langle\rangle \vdash c : t}$$

$$\text{exchg}\ \frac{\Gamma_1, x : \sigma_1,\ y : \sigma_2,\ \Gamma_2 \mathrel{?} \mathsf{R} \times \langle \mathsf{a,b} \rangle \times \mathsf{S} \vdash e : \tau}{\Gamma_1,\ y : \sigma_2, x : \sigma_1,\ \Gamma_2 \mathrel{?} \mathsf{R} \times \langle \mathsf{b,a} \rangle \times \mathsf{S} \vdash e : \tau}$$

Example: reuse bounds

$$(\lambda v.x{+}v{+}v)\ (x{+}y)$$

$$\dfrac{(\text{app})\ \dfrac{(\text{abs})\ \dfrac{x:\mathbb{Z}, v:\mathbb{Z}?\langle 1,2\rangle \vdash x+v+v:\mathbb{Z}}{x:\mathbb{Z}?\langle 1\rangle \vdash (\lambda v.x+v+v):\mathbb{Z}\xrightarrow{2}\mathbb{Z}} \qquad \dfrac{\vdots}{x':\mathbb{Z}, y:\mathbb{Z}?\langle 1,1\rangle \vdash x'+y:\mathbb{Z}}}{(\equiv)\ \dfrac{x:\mathbb{Z}, x':\mathbb{Z}, y:\mathbb{Z}?\langle 1\rangle \times (2 * \langle 1,1\rangle) \vdash (\lambda v.x+v+v)\ (x'+y):\mathbb{Z}}{(\text{contr})\ \dfrac{x:\mathbb{Z}, x':\mathbb{Z}, y:\mathbb{Z}?\langle 1,2,2\rangle \vdash (\lambda v.x+v+v)\ (x'+y):\mathbb{Z}}{x:\mathbb{Z}, y:\mathbb{Z}?\langle 3,2\rangle \vdash (\lambda v.x+v+v)\ (x+y):\mathbb{Z}}}}{}$$

$$(\lambda v.x+v+v)\ (x+y) \ \rightsquigarrow_\beta\ x+(x+y)+(x+y)$$

$$x:\mathbb{Z}, y:\mathbb{Z}?\langle 3,2\rangle \vdash x+(x+y)+(x+y):\mathbb{Z}$$

Example: resources

- Per context

- Tracks resource requirements, e.g.

$$\varnothing ? \{\text{gps}\} \vdash \textbf{access gps} : \text{Coord}$$

- "impure" lambda

$$\varnothing ? \{\text{gps}\} \vdash (\text{fn } x \,.\, \textbf{access gps} + x) : \text{Int} \xrightarrow{\{\}} \text{Coord}$$

gps at declaration site

$$\varnothing ? \{\} \vdash (\text{fn } x \,.\, \textbf{access gps} + x) : \text{Int} \xrightarrow{\{\text{gps}\}} \text{Coord}$$

gps at call site

Example: resources

$$\text{abs}\ \frac{\Gamma,\ x : \sigma \ ?\ \mathsf{R} \cup \mathsf{S} \vdash e : \tau}{\Gamma \ ?\ \mathsf{R} \vdash \lambda x.e : \sigma \xrightarrow{\mathsf{S}} \tau} \qquad \text{var}\ \frac{}{x : \sigma \ ?\ \varnothing \vdash x : \sigma}$$

$$\text{app}\ \frac{\Gamma_1 \ ?\ \mathsf{R} \vdash e_1 : \sigma \xrightarrow{\mathsf{T}} \tau \qquad \Gamma_2 \ ?\ \mathsf{S} \vdash e_2 : \sigma}{\Gamma_1, \Gamma_2 \ ?\ \mathsf{R} \cup \mathsf{S} \cup \mathsf{T} \vdash e_1\ e_2 : \tau}$$

$$\text{contr}\ \frac{\Gamma_1,\ x : \sigma,\ y : \sigma,\ \Gamma_2 \ ?\ \mathsf{R} \vdash e : \tau}{\Gamma_1,\ z : \sigma,\ \Gamma_2 \ ?\ \mathsf{R} \vdash e\,[z \leftarrow x, z \leftarrow y] : \tau}$$

$$\text{access}\ \frac{\mathbf{R} : \tau}{\Gamma,\ x : \tau \ ?\ \{\mathbf{R}\} \vdash \textbf{access R} : \tau} \qquad \text{sub}\ \frac{\Gamma \ ?\ \mathsf{R} \vdash e : \tau \qquad \mathsf{R} \subseteq \mathsf{R'}}{\Gamma \ ?\ \mathsf{R'} \vdash e : \tau}$$

Example: resources

$$\text{abs}\ \frac{\Gamma, x : \sigma \mathbin{?} \mathsf{R} \cup \mathsf{S} \vdash e : \tau}{\Gamma \mathbin{?} \mathsf{R} \vdash \lambda x.e : \sigma \xrightarrow{\mathsf{S}} \tau}$$

introduces non-determinism

$$\varnothing ? \{\mathsf{gps}\} \vdash (\mathsf{fn}\ x\ .\ \mathbf{access}\ \mathsf{gps} + x)\ :\ \mathrm{Int} \xrightarrow{\{\}} \mathrm{Coord}$$ declaration site

$$\varnothing ? \{\} \vdash (\mathsf{fn}\ x\ .\ \mathbf{access}\ \mathsf{gps} + x)\ :\ \mathrm{Int} \xrightarrow{\{\mathsf{gps}\}} \mathrm{Coord}$$ call site

$$\text{signature}\ \frac{\Gamma \mathbin{?} \mathsf{R} \vdash e : \tau}{\Gamma \mathbin{?} \mathsf{R} \vdash (e : \tau) : \tau}$$

Example: resources

- Implicit parameters (in Haskell) are an example
- Type class constraints are related

$$\text{abs}\ \frac{\Gamma, x : \sigma \mathrel{?} \mathsf{R} \vdash e : \tau}{\Gamma \mathrel{?} \mathsf{R} \vdash \lambda x.e : \sigma \to \tau}$$

Example: resources

$$\text{abs}\ \frac{\Gamma,\ x : \sigma \ ?\ \mathsf{R} \cup \mathsf{S} \vdash e : \tau}{\Gamma\ ?\ \mathsf{R} \vdash \lambda x.e : \sigma \xrightarrow{\mathsf{S}} \tau}
\qquad
\text{var}\ \frac{}{x : \sigma\ ?\ \varnothing \vdash x : \sigma}$$

$$\text{app}\ \frac{\Gamma_1\ ?\ \mathsf{R} \vdash e_1 : \sigma \xrightarrow{\mathsf{T}} \tau \qquad \Gamma_2\ ?\ \mathsf{S} \vdash e_2 : \sigma}{\Gamma_1, \Gamma_2\ ?\ \mathsf{R} \cup \mathsf{S} \cup \mathsf{T} \vdash e_1\ e_2 : \tau}$$

$$\text{contr}\ \frac{\Gamma_1,\ x : \sigma,\ y : \sigma,\ \Gamma_2\ ?\ \mathsf{R} \vdash e : \tau}{\Gamma_1,\ z : \sigma,\ \Gamma_2\ ?\ \mathsf{R} \vdash e\,[z \leftarrow x, z \leftarrow y] : \tau}$$

$$\text{weak}\ \frac{\Gamma\ ?\ \mathsf{R} \vdash e : \tau}{\Gamma,\ x : \sigma\ ?\ \mathsf{R} \vdash e : \tau}
\qquad
\text{sub}\ \frac{\Gamma\ ?\ \mathsf{R} \vdash e : \tau \qquad \mathsf{R} \subseteq \mathsf{R'}}{\Gamma\ ?\ \mathsf{R'} \vdash e : \tau}$$

Example: reuse bounds

$$\text{abs}\ \frac{\Gamma,\ x : \sigma \mathbin{?} \mathsf{R} \times \langle \mathsf{s} \rangle \vdash e : \tau}{\Gamma \mathbin{?} \mathsf{R} \vdash \lambda x.e : \sigma \xrightarrow{\mathsf{s}} \tau}$$

$$\text{var}\ \frac{}{x : \sigma \mathbin{?} \langle 1 \rangle \vdash x : \sigma}$$

$$\text{app}\ \frac{\Gamma_1 \mathbin{?} \mathsf{R} \vdash e_1 : \sigma \xrightarrow{\mathsf{t}} \tau \qquad \Gamma_2 \mathbin{?} \mathsf{S} \vdash e_2 : \sigma}{\Gamma_1, \Gamma_2 \mathbin{?} \mathsf{R} \times (\mathsf{t} * \mathsf{S}) \vdash e_1\, e_2 : \tau}$$

$$\text{contr}\ \frac{\Gamma_1,\ x : \sigma,\ y : \sigma,\ \Gamma_2 \mathbin{?} \mathsf{R} \times \langle \mathsf{a,b} \rangle \times \mathsf{S} \vdash e : \tau}{\Gamma_1,\ z : \sigma,\ \Gamma_2 \mathbin{?} \mathsf{R} \times \langle \mathsf{a+b} \rangle \times \mathsf{S} \vdash e\,[z \leftarrow x, z \leftarrow y] : \tau}$$

$$\text{weak}\ \frac{\Gamma \mathbin{?} \mathsf{R} \vdash e : \tau}{\Gamma,\ x : \sigma \mathbin{?} \mathsf{R} \times \langle 0 \rangle \vdash e : \tau}$$

$$\text{sub}\ \frac{\Gamma \mathbin{?} \mathsf{R} \vdash e : \tau \qquad \mathsf{R} \leq \mathsf{R'}}{\Gamma \mathbin{?} \mathsf{R'} \vdash e : \tau}$$

combine per-variable and per-context

Coeffect calculus

$$\text{abs}\ \frac{\Gamma, x : \sigma \ ?\ \mathsf{R} \ltimes \langle \mathsf{s} \rangle \vdash e : \tau}{\Gamma \ ?\ \mathsf{R} \vdash \lambda x.e : \sigma \xrightarrow{\mathsf{s}} \tau}$$

$$\text{var}\ \frac{}{x : \sigma \ ?\ \langle \mathsf{use} \rangle \vdash x : \sigma}$$

$$\text{app}\ \frac{\Gamma_1 \ ?\ \mathsf{R} \vdash e_1 : \sigma \xrightarrow{\mathsf{t}} \tau \qquad \Gamma_2 \ ?\ \mathsf{S} \vdash e_2 : \sigma}{\Gamma_1, \Gamma_2 \ ?\ \mathsf{R} \rtimes (\mathsf{t} \circledast \mathsf{S}) \vdash e_1\, e_2 : \tau}$$

$$\text{contr}\ \frac{\Gamma_1, x : \sigma, y : \sigma, \Gamma_2 \ ?\ \mathsf{R} \ltimes \langle \mathsf{a} \rangle \ltimes \langle \mathsf{b} \rangle \ltimes \mathsf{S} \vdash e : \tau}{\Gamma_1, z : \sigma, \Gamma_2 \ ?\ \mathsf{R} \rtimes \langle \mathsf{a} \oplus \mathsf{b} \rangle \rtimes \mathsf{S} \vdash e\,[z \leftarrow x, z \leftarrow y] : \tau}$$

$$\text{weak}\ \frac{\Gamma \ ?\ \mathsf{R} \vdash e : \tau}{\Gamma, x : \sigma \ ?\ \mathsf{R} \rtimes \langle \mathsf{ign} \rangle \vdash e : \tau}$$

$$\text{sub}\ \frac{\Gamma \ ?\ \mathsf{R} \vdash e : \tau \qquad \mathsf{R} \leq \mathsf{R'}}{\Gamma \ ?\ \mathsf{R'} \vdash e : \tau}$$

Algebraic structure

- Coeffect scalar $(C, \circledast, \oplus, \mathsf{use}, \mathsf{ign}, \leq)$
  - $\circledast$ - sequential compose
  - $\oplus$ - share (contraction)
  - use - variables
  - ign - null
  - $\leq$ - subcoeffecting

core (almost semi-ring)

- Coeffect shapes $(S, [-], \diamond, \underline{0}, \underline{1})$
  - $[-]$ - context to shape
  - $\diamond$ - compose shapes
  - $\underline{0}$ - empty
  - $\underline{1}$ - singleton
- Coeffect algebra $(\ltimes, \rtimes, \perp)$
  - $\ltimes : C^n \times C^m \rightarrow C^{n \diamond m}$ - merge coeffects in premise
  - $\rtimes : C^n \times C^m \rightarrow C^{n \diamond m}$ - merge coeffects in conclusion
  - $\perp : C^{\underline{0}}$ - empty context

unified

(derived
- $\langle\,\rangle : C \rightarrow C^{\underline{1}}$ - lift to shaped
- $\circledast : C \times C^m \rightarrow C^m$ - scalar-shaped compose

)

combine per-variable and per-context

Coeffect calculus

$$\text{abs}\ \frac{\Gamma, x : \sigma \,?\, \mathsf{R} \ltimes \langle \mathsf{s} \rangle \vdash e : \tau}{\Gamma \,?\, \mathsf{R} \vdash \lambda x.e : \sigma \xrightarrow{\mathsf{s}} \tau}
\qquad
\text{var}\ \frac{}{x : \sigma \,?\, \langle \mathsf{use} \rangle \vdash x : \sigma}$$

$$\text{app}\ \frac{\Gamma_1 \,?\, \mathsf{R} \vdash e_1 : \sigma \xrightarrow{\mathsf{t}} \tau \qquad \Gamma_2 \,?\, \mathsf{S} \vdash e_2 : \sigma}{\Gamma_1, \Gamma_2 \,?\, \mathsf{R} \rtimes (\mathsf{t} \circledast \mathsf{S}) \vdash e_1\, e_2 : \tau}$$

$$\text{contr}\ \frac{\Gamma_1, x : \sigma, y : \sigma, \Gamma_2 \,?\, \mathsf{R} \ltimes \langle \mathsf{a} \rangle \ltimes \langle \mathsf{b} \rangle \ltimes \mathsf{S} \vdash e : \tau}{\Gamma_1, z : \sigma, \Gamma_2 \,?\, \mathsf{R} \rtimes \langle \mathsf{a} \oplus \mathsf{b} \rangle \rtimes \mathsf{S} \vdash e\,[z \leftarrow x, z \leftarrow y] : \tau}$$

$$\text{weak}\ \frac{\Gamma \,?\, \mathsf{R} \vdash e : \tau}{\Gamma, x : \tau \,?\, \mathsf{R} \rtimes \langle \mathsf{ign} \rangle \vdash e : \tau}
\qquad
\text{sub}\ \frac{\Gamma \,?\, \mathsf{R} \vdash e : \tau \qquad \mathsf{R} \leq \mathsf{R'}}{\Gamma \,?\, \mathsf{R'} \vdash e : \tau}$$

Comparing coeffects & effects

$$\frac{\Gamma, x : \sigma \mathbin{?} \mathsf{R} \ltimes \langle \mathsf{s} \rangle \vdash e : \tau}{\Gamma \mathbin{?} \mathsf{R} \vdash \lambda x.e : \sigma \xrightarrow{\mathsf{s}} \tau}$$

$$\frac{\Gamma, x : \sigma \vdash e : \tau, \mathsf{F}}{\Gamma \vdash \lambda x \,.\, e : \sigma \xrightarrow{\mathsf{F}} \tau, \emptyset}$$

$$\frac{\Gamma, x : \sigma \mathbin{?} \mathsf{R} \times \langle \mathsf{s} \rangle \vdash e : \tau}{\Gamma \mathbin{?} \mathsf{R} \vdash \lambda x.e : \sigma \xrightarrow{\mathsf{s}} \tau}$$

reuse bounds

$$\frac{\Gamma, x : \sigma \mathbin{?} \mathsf{R} \cup \mathsf{S} \vdash e : \tau}{\Gamma \mathbin{?} \mathsf{R} \vdash \lambda x.e : \sigma \xrightarrow{\mathsf{S}} \tau}$$

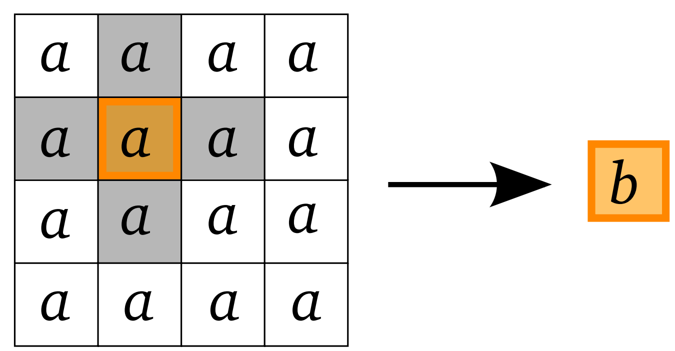
resources

Examples

	C	seq $\circledast$	share $\oplus$	var use	weak ign	sub $\leq$	split $\ltimes$	merge $\rtimes$	empty $\perp$
BLL	$\mathbb{N}$	$*$	$+$	1	0	$\leq$	$\times$	$\times$	$\langle\rangle$
liveness	$\mathbb{B}$	$\wedge$	$\vee$	T	F	F $\leq$ T	$\times$	$\times$	$\langle\rangle$
dataflow	$\mathbb{N}$	$+$	max	0	0	$\leq$	$\times$	$\times$	$\langle\rangle$
implicit	$v : t$	$\cup$	$\cup$	$\varnothing$	$\varnothing$	$\subseteq$	$\cup$	$\cup$	$\varnothing$
count	$\mathbb{N}$	$+$	$+$	0	0	$\leq$	$+$	$+$	0

Coeffects in practice: Ypnos

- Ypnos: language for scientific computing
  - Based on sub-language of "stencils"

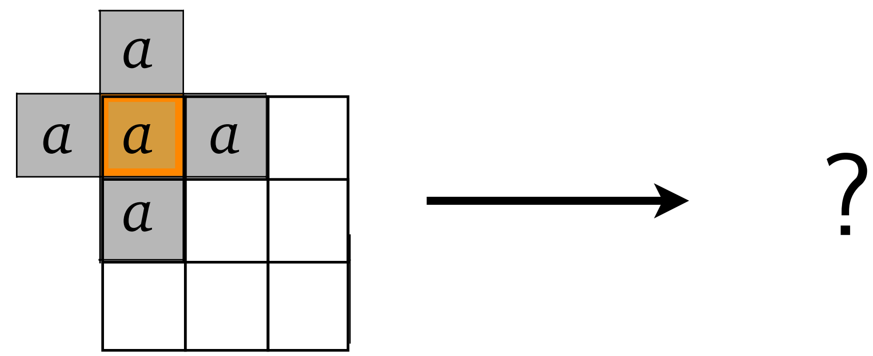


  - Parameterisable by different data structures
  - Coeffect analysis tracks data access
- Various safety and optimisation guarantees

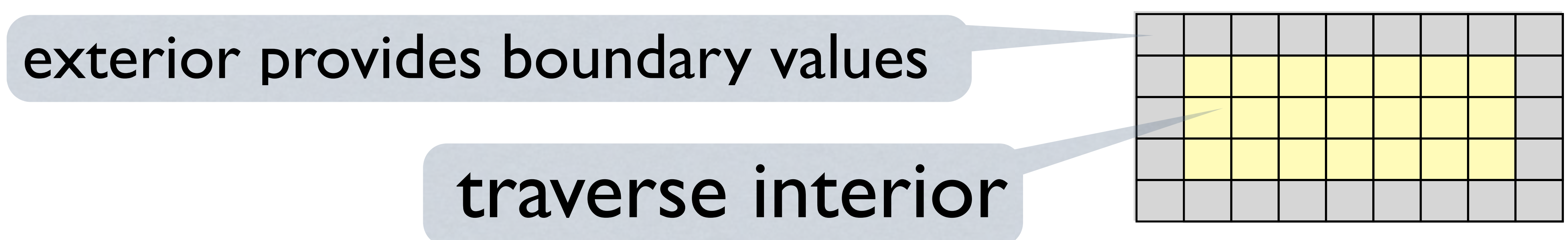
(Orchard, Mycroft - 2010)

Ypnos

- At boundaries?



- Solution: exterior elements

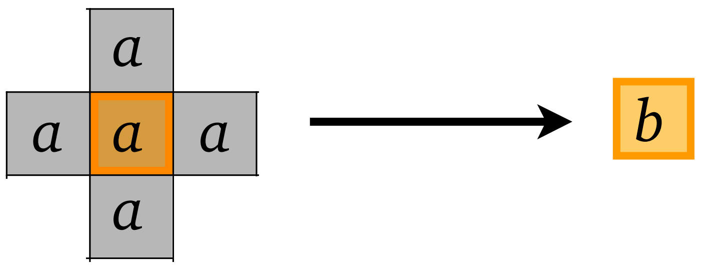


- Requires array with adequate exterior

Ypnos

- 5-point stencil (local mean)

```
f(A, i, j) =
  (A(i-1,j) + A(i+1,j)  +
   A(i,j-1) + A(i, j+1) +
   A(i, j)) / 5
```



local operation, at some context

- Relative data access as a coeffect, e.g.:

$$\mathrm{A} : \mathrm{Float} \; ? \; \{(0,0),(-1,0),(1,0),(0,1),(0,-1)\} \vdash$$
$$(\texttt{A[0][0] + A[-1][0] +}$$
$$\texttt{A[1][0] + A[0][1] + A[0][-1])/5.0} \; : \mathrm{Float}$$

Ypnos

- Two-level language
  - Haskell (outer) $\Gamma \vdash e : \mathrm{t}$
  - Ypnos stencils (inner) $\Gamma; \Gamma' ? \mathsf{r} \vdash e : \mathrm{t}$
- Interface between the two:

data access

$$\frac{\Gamma;\ x : \mathrm{t}_1\ ?\ \mathsf{r} \vdash \mathrm{e} : \mathrm{t}_2}{\Gamma \vdash (\textsf{stencil}\ x\ .\ \mathrm{e}) : \mathrm{Array}\ \mathsf{r}\ \mathrm{t}_1 \rightarrow \mathrm{t}_2}$$

exterior size

Ypnos - Grid Patterns

- Array access only by pattern matching
- e.g.

~~stencil `A . (A[0][0] + A[-1][0] + A[1][0] + A[0][1] + A[0][-1]) /5.0`~~

```
stencil  | _  t _ |
         | l @c r |
         | _  b _ | . (t + l + c + r + b)/5.0
```

- Patterns are static => decidable coeffects

Semantics…

- Effect calculus
  - Indexed strong monad — sequential composition + plumbing
- Coeffect calculus
  - Indexed comonad — sequential composition
  - Indexed structural comonad — + plumbing
  - Indexed lax & colax monoidal structure — context merge/split

recall

Parametric effect monads

$$\text{comp}\ \frac{f : X \to \mathbf{M}_{F} Y \qquad g : Y \to \mathbf{M}_{G} Z}{(f;g)\ : X \to \mathbf{M}_{(F \sqcup G)} Z}$$

$$\text{id}\ \frac{}{id_A\ : A \to \mathbf{M}_{\varnothing} A}$$

monoid $(F, \sqcup, \varnothing)$

$$[\ \Gamma \vdash\ e\ :\ \tau,\ F\ ]\ :\ [\Gamma] \to \mathbf{M}_F[\tau]$$

$$[\sigma \xrightarrow{F} \tau] = [\sigma] \to \mathbf{M}_F[\tau]$$

[see Katsumata, POPL 2014]

Indexed comonad (parametric coeffect comonad)

$$\text{comp}\ \frac{f : \mathsf{C}_{\mathsf{r}} X \to Y \qquad\qquad g : \mathsf{C}_{\mathsf{s}} Y \to Z}{(f;g)\ : \mathsf{C}_{\mathsf{r} \circledast \mathsf{s}} X \to Z}$$

$$\text{id}\ \frac{}{id_A\ : \mathsf{C}_{\mathsf{use}} A \to A}$$

monoid $(\mathsf{R}, \circledast, \mathsf{use})$

$[\ \Gamma\ ?\ \mathsf{r} \vdash\ e\ :\ \tau\ ]\ :\ \mathsf{C}^{\mathsf{l}}{}_{\mathsf{r}}[\Gamma] \to [\tau]$

$[\sigma \xrightarrow{\mathsf{r}} \tau] = \mathsf{C}_{\mathsf{r}}[\sigma] \to [\tau]$

[see Petricek, Orchard, Mycroft, ICALP 2013, ICFP 2014]

Indexed structural comonad

$$[\, \Gamma \,?\, R \vdash e : \tau \,] : C^{|\Gamma|}_{R}[\Gamma] \to [\tau]$$

$$\text{comp}\ \frac{f : C^{n}_{R}(X_1,..X_n) \to Y \quad g : C^{1}_{s} Y \to Z}{(f;g) \ : C^{n}_{s \circledast R}(X_1,..,X_n) \to Z}$$

$$\text{id-left}\ \frac{}{id_A \ : C^{0}_{\perp} A \to A} \qquad \text{id-right}\ \frac{}{id_A \ : C^{1}_{use} A \to A}$$

monoid left-action $(R^n, \circledast)$

[see Petricek, Orchard, Mycroft, ICFP 2014]

Merging and splitting

$$\text{merge}\ \frac{f : \mathbf{C}^{n}{}_{R}(X_1,..,X_n) \qquad g : \mathbf{C}^{m}{}_{S}(Y_1,..,Y_n)}{(\text{merge}\ f\ g)\ : \mathbf{C}^{n\diamond m}{}_{R\ltimes S}(X_1,..,X_n, Y_1,..,Y_n)}$$

$$\text{split}\ \frac{f : \mathbf{C}^{n\diamond m}{}_{R\rtimes S}(X_1,..,X_n, Y_1,..,Y_n)}{(\text{split}\ f)\ :\ \mathbf{C}^{n}{}_{R}(X_1,..,X_n) \times \mathbf{C}^{m}{}_{S}(Y_1,..,Y_n)}$$

What's next

- Bieffects $\Gamma \ ? \ \mathsf{R} \vdash e : \tau, \ \mathsf{F}$
- Extend to other calculi (pi-calculus?)
- Other shapes
- Extend to program logics (partial operations)
- New examples: security? information flow?
- Reused semantic derivation techniques for other types

Conclusions

- Dualising effects is subtle due to asymmetry
- Coeffects require more algebraic structure
- Lots of interesting examples being discovered

- Lots of new work:
  - Brunel, Gaboardi, Mazza, Zdancewic "*A core quantitative coeffect calculus*" (ESOP 2014)
  - Ghica, Smith "*Bounded linear types in a resource semiring*" (ESOP 2014)
  - Petricek, Orchard, Mycroft "*Coeffects: a calculus of context-dependent computation*" (ICFP 2014)
  - Orchard, Petricek "*Embedding effect systems in Haskell*" (Haskell, 2014)

Thanks!

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upcoming thesis “Context-aware programming languages”