Complexity bounds from abstract categorical models of containers

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Complexity bounds and optimisations

Definition

Given two programs f and g which are equivalent ($f \equiv g$) then the rewrite $f \rightsquigarrow g$ is an optimisation iff:

$$[g]_n \in \mathcal{O}([f]_n)$$

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The axioms of useful categorical structures imply general optimisations.

i.e., an axiom $f \equiv g$ can be *oriented* $f \rightsquigarrow g$ which is guaranteed to not make the program asymptotically slower.

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Object mapping is data type $[\,] : * \to *$ and morphism mapping $map : \forall a, b.(a \to b) \to ([a] \to [b])$.

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- with two axioms:

$$[\text{F1}]\ \ \mathsf{F}id_A \equiv id_{\mathsf{F}A} \qquad\qquad [\text{F2}]\ \ \mathsf{F}(g \circ f) \equiv \mathsf{F}g \circ \mathsf{F}f$$

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- Can model element-wise (point-wise) data structure traversals

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A: Depends on input "size"? What is the size of an $\mathsf{F}A$?

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Useful: **functor lifting produces a size-preserving function**

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A slight refinement...

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For any discretely finite container F*, the morphism mapping operation has lower bound complexity:*

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$\big(n[\Omega(m)]$ is a structure of size n with elements at least size $m\big)$

Upper bounds are more useful

Proposition

There exists terms P_n and $Q_n \geq 1$, parameterised by n, such that:

$$[\mathsf{F}f]_{n[\mathcal{O}(m)]} \in \mathcal{O}(P_n + n\, Q_n[f]_m) \tag{1}$$

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Follows from lower-bound: at least n uses of f (at size at most m) with possible additional overhead:

- P_n accounts for time traversing the container to reach the leaves (the elements) and
- Q_n accounts for any extraneous applications of f beyond the linear (in n) use.

□

For naturality

Given two functors F, G and natural transformation $\eta_A : \mathsf{F}A \to \mathsf{G}A$:

$$
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Therefore, if $n \in \mathcal{O}(k(n))$ then $(\mathsf{G}f \circ \eta_A) \rightsquigarrow (\eta_B \circ \mathsf{F}f)$, otherwise the converse.

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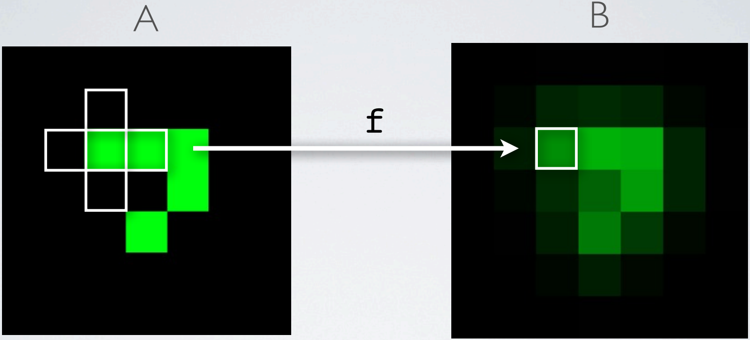
- But suggestion that $\mathsf{F}f \circ \uparrow_A \rightsquigarrow \uparrow_B \circ f$.

Stencil Computations




```
for (i=0; i<N; i++)
    for (j=0; j<M; j++)
        B[i][j] = f(A[i][j],   A[i-1][j], A[i+1][j],
                     A[i][j-1], A[i][j+1]);
```

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$$\textbf{functor}\ \frac{f : A \to B}{\mathsf{F}f : \mathsf{F}A \to \mathsf{F}B} \qquad \textbf{comonad}\ \frac{g : \mathsf{F}A \to B}{g^{\dagger} : \mathsf{F}A \to \mathsf{F}B}$$

Example comonad: Array

Array is an array with a *cursor*

a	a	a	a
a	$\boxed{a}$	a	a
a	a	a	a
a	a	a	a

$$f : \mathbf{Array}\, a \rightarrow b$$

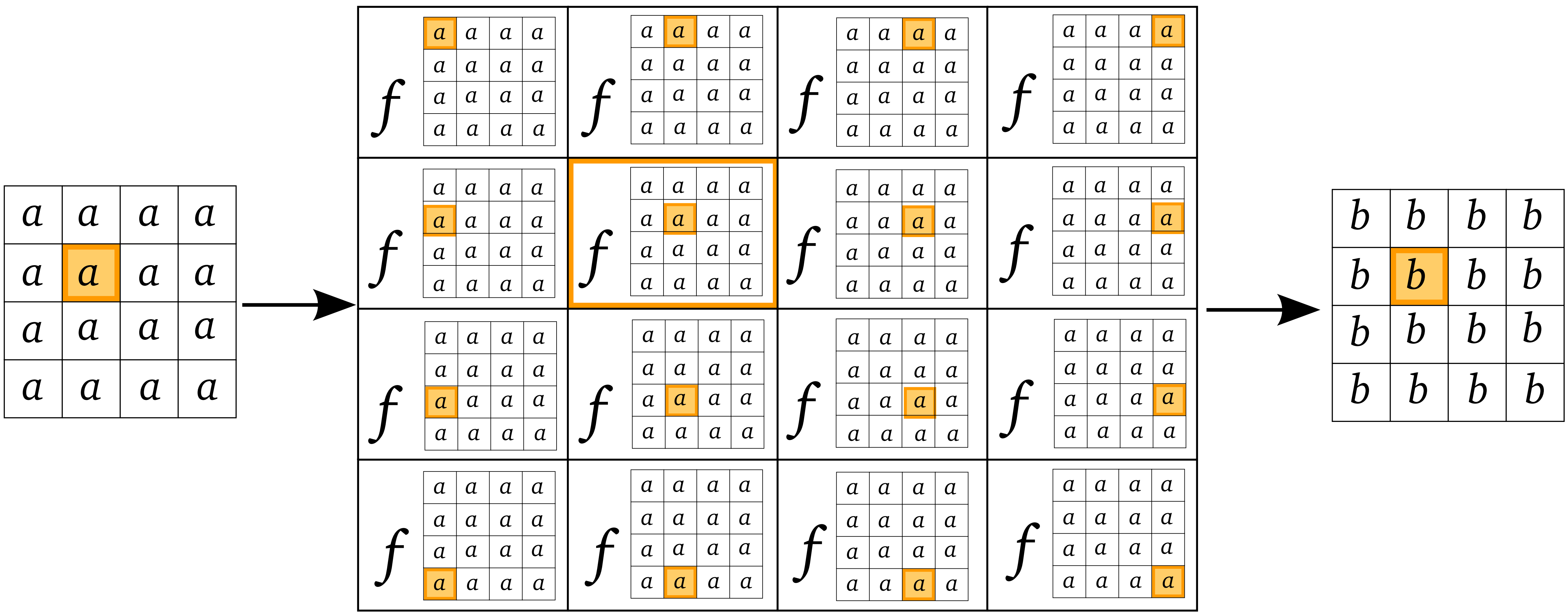
a	a	a	a
a	$\boxed{a}$	a	a
a	a	a	a
a	a	a	a

$$\longrightarrow \boxed{b}$$

[see "Ypnos: Declarative, Parallel Structured Grid Programming", Orchard, Bolingbroke, Mycroft'10]

Example comonad: Array

$$(\_)^{\dagger} : (\mathbf{Array}\, a \to b) \to (\mathbf{Array}\, a \to \mathbf{Array}\, b)$$



Generalised-map on arrays (e.g. convolution)

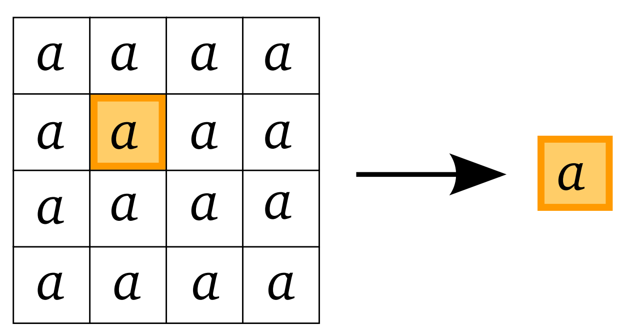
Comonads

(Co)unit

$$\epsilon : Da \rightarrow a$$

Extract the value at the "current context"

$$\epsilon : \mathbf{Array}\, a \rightarrow a$$



Comonads

[C1] $\epsilon^{\dagger} = id$

[C2] $\epsilon \circ f^{\dagger} = f$

[C3] $(g \circ f^{\dagger})^{\dagger} = g^{\dagger} \circ f^{\dagger}$

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- *extract* operation $\varepsilon_A : \mathsf{F}A \to A$
- with three axioms:

$$[\mathrm{C1}]\ \ \varepsilon^{\dagger} \equiv id \qquad [\mathrm{C2}]\ \ \varepsilon \circ f^{\dagger} \equiv f \qquad [\mathrm{C3}]\ \ g^{\dagger} \circ f^{\dagger} \equiv (g \circ f^{\dagger})^{\dagger}$$

Axiom [C3], associativity

$$\dfrac{\dfrac{f : \mathsf{F}A \to B}{f^{\dagger} : \mathsf{F}A \to \mathsf{F}B} \qquad \dfrac{g : \mathsf{F}B \to C}{g^{\dagger} : \mathsf{F}B \to \mathsf{F}C}}{g^{\dagger} \circ f^{\dagger} : \mathsf{F}A \to \mathsf{F}C} \equiv$$

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Compare:

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for i = 1..n
 for j = 1 .. m
   B(i, j) = f (A, i, j)
for i = 1 .. n
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$$\dfrac{\dfrac{f : \mathsf{F}A \to B}{f^\dagger : \mathsf{F}A \to \mathsf{F}B} \quad \dfrac{g : \mathsf{F}B \to C}{g^\dagger : \mathsf{F}B \to \mathsf{F}C}}{g^\dagger \circ f^\dagger : \mathsf{F}A \to \mathsf{F}C} \equiv \dfrac{\dfrac{\dfrac{f : \mathsf{F}A \to B}{f^\dagger : \mathsf{F}A \to \mathsf{F}B} \quad g : \mathsf{F}B \to C}{g \circ f^\dagger : \mathsf{F}A \to C}}{(g \circ f^\dagger)^\dagger : \mathsf{F}A \to \mathsf{F}C}$$

Compare:

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for i = 1..n
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Q: Is $(g \circ f^\dagger)^\dagger \rightsquigarrow g^\dagger \circ f^\dagger$ always asymptotically better?

Implicit complexity of (container) comonads

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Pass (asympotitcally larger) $x : \mathsf{F}A$ to $f : \mathsf{F}A \to B$ at each context.

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- By $[C3], [C1]$, apply the above at every context:

$$(\varepsilon_{\mathbb{N}} \circ \mathsf{size}^\dagger_A)^\dagger \overset{[C3]}{\equiv} \varepsilon^\dagger_{\mathbb{N}} \circ \mathsf{size}^\dagger_A \overset{[C1]}{\equiv} \mathsf{size}^\dagger_A$$

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Therefore:

Proposition

Axiom [C3] can be oriented as $(g \circ f^\dagger)^\dagger \rightsquigarrow g^\dagger \circ f^\dagger$ *guaranteeing an asymptotic improvement.*

Proof.

From the above:

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[(g \circ f^\dagger)^\dagger]_n &\in \mathcal{O}(P_n + nQ_n([g]_n + P_n + nQ_n[f]_n)) \\
&\in \mathcal{O}(P_n + nQ_n[g]_n + (nQ_n)^2[f]_n + nQ_nP_n)
\end{aligned}$$

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implies $[f^{\dagger}]_n \in \mathcal{O}(P_n + n[f]_n)$