Foundations for Tracing Functional Programs and the Correctness of Algorithmic Debugging

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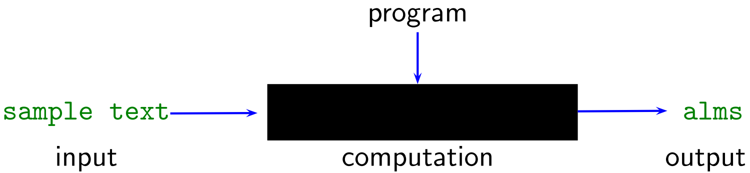
26th April 2006

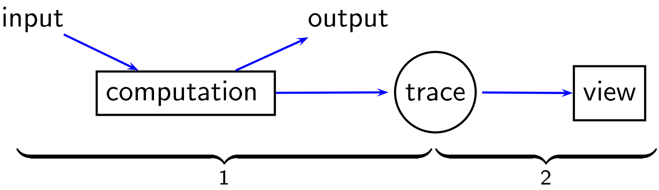
Why Tracing?

```
insert :: Ord a => a -> [a] -> [a]

insert x [] = [x]
insert x (y:ys) =
  if x > y then y : insert x ys
           else x : ys

sort :: Ord a => [a] -> [a]
sort [] = []
sort (x:xs) = insert x (sort xs)

main = getLine >>= putStrLn . sort
```



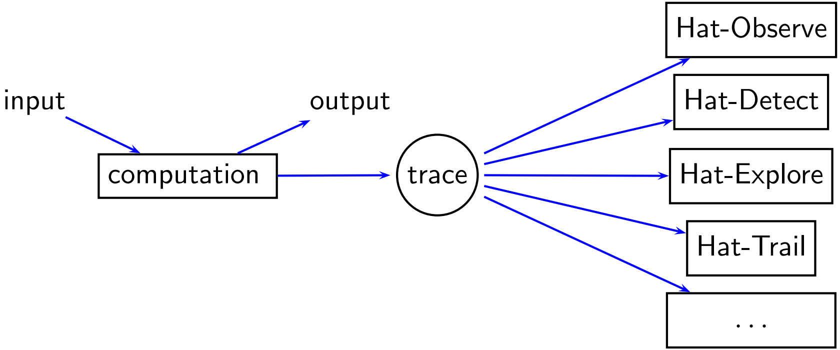
- Locate a fault (wrong output, run-time error, non-termination).
- Comprehend a program.

Two-Phase Tracing: A Trace as Data Structure



- Liberates from time arrow of computation.
- Enables views based on different execution models.
  (small-step, big-step, interpreter with environment, denotational)
- Enables compositional views.

- Multi-View Tracer



- Trace = Augmented Redex Trail (ART); distilled as unified trace.

Aim: A theoretical model of this trace and its views.

Overview

1. Definition of the Trace through Graph Rewriting
2. Properties of the Trace
3. Views of the Trace
    - Observation of Functions
    - Following Redex Trails
    - Algorithmic Debugging
4. Correctness of Algorithmic Debugging
5. Future Work & Summary

The Programming Language

Launchbury's and related semantics

- Subset of λ-calculus plus `case` for matching.
- Any program can be translated into this core calculus.

For tracing

- Close relationship between trace and original program essential.
- Language must have most frequently used features:
  - named functions
  - pattern matching

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- Subset of λ-calculus plus `case` for matching.
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For tracing

- Close relationship between trace and original program essential.
- Language must have most frequently used features:
  - named functions
  - pattern matching

$\Rightarrow$ Higher-order term rewriting system

```
sort [] = []                              or   sort = foldr insert []
sort (x:xs) = insert x (sort xs)

insert x [] = [x]
insert x (y:ys) = if x > y then y : (insert x ys) else x : ys
```

What is a Good Trace?

Program + input determine every detail of computation.

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What is a computation? Semantics answers:

- Term rewriting: A sequence of expressions.
  $t_1 \longrightarrow t_2 \longrightarrow t_3 \longrightarrow t_4 \longrightarrow t_5 \longrightarrow \ldots \longrightarrow t_n$
- Natural semantics: A proof tree.

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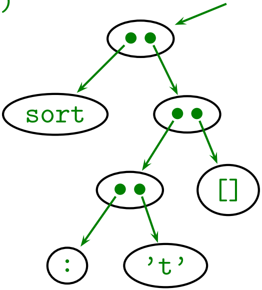
But

- Lots of redundancy.
- Much structure already lost.

Graph Rewriting I

`sort ('t':[])`

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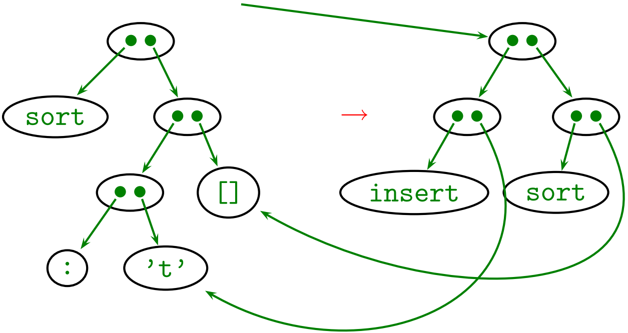


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- Create new nodes for right-hand-side.
- Nodes of subexpressions are shared.

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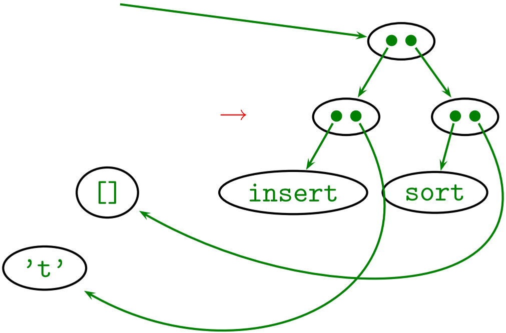


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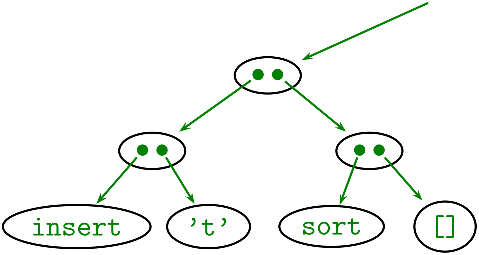
- Create new nodes for right-hand-side.
- Nodes of subexpressions are shared.
- Some old nodes become garbage.

Graph Rewriting II

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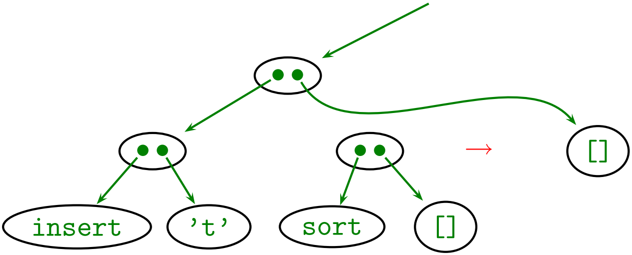


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- Application node of redex replaced by new node.

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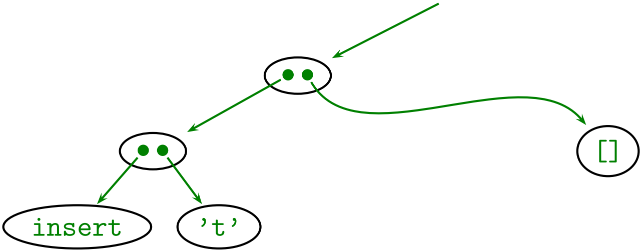


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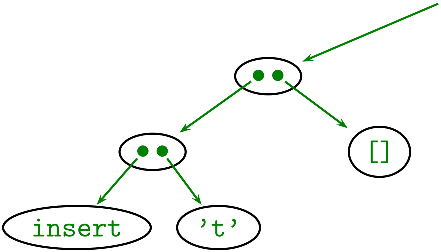


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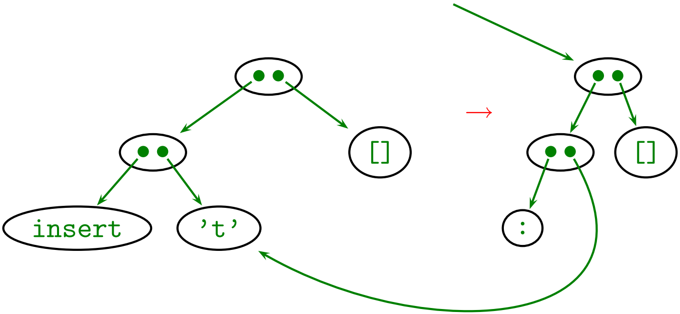


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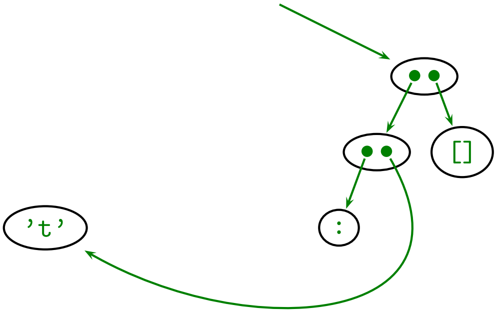


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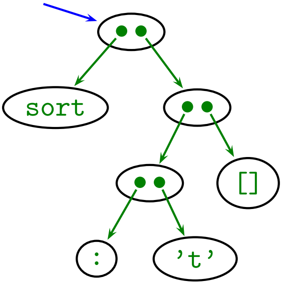
The Trace



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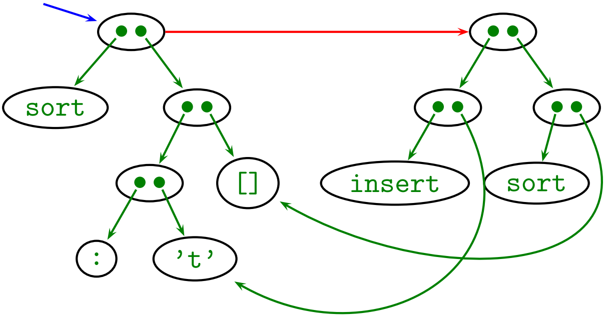

- New nodes for right-hand-side, connected via result pointer.
- Only add to graph, never remove.
- Sharing ensures compact representation.

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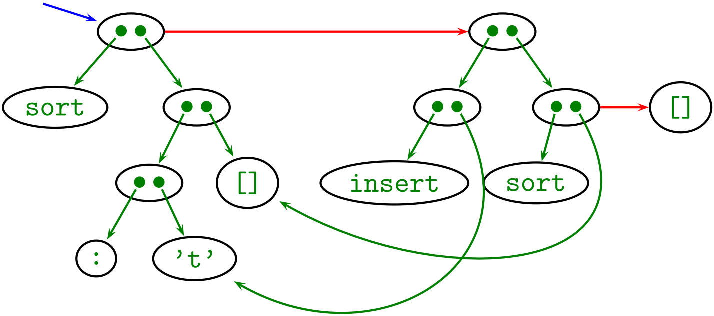

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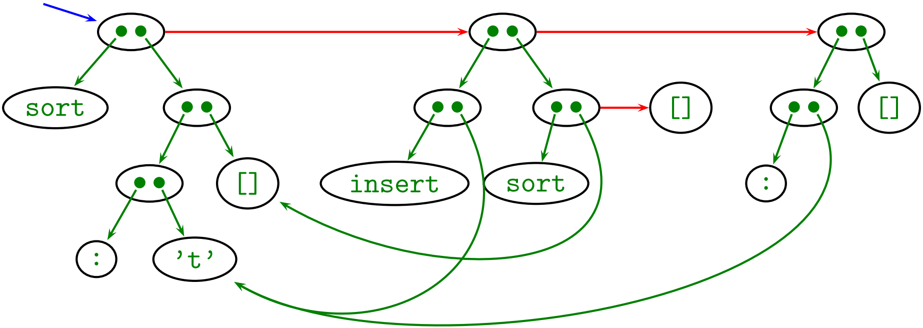

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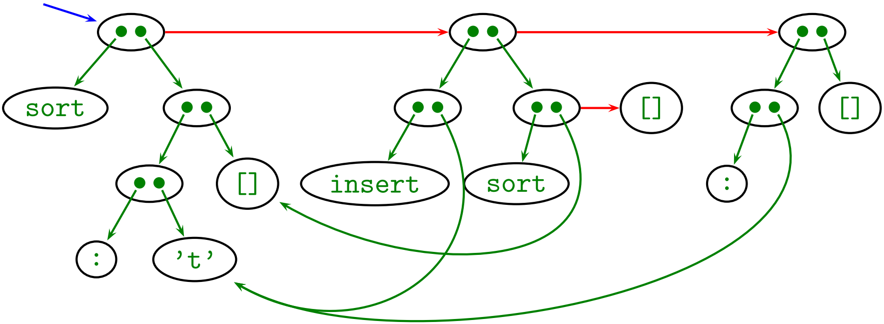
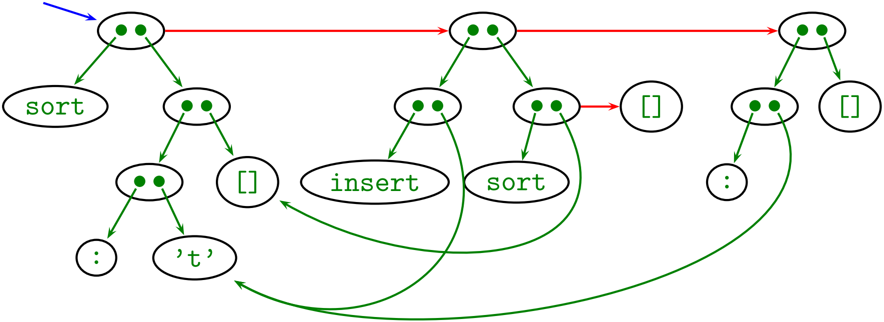
The Node Labels



term constructor	T	$:=$	a	atom
		$\mid$	$n\ m$	application of nodes
atom	a	$:=$	$f \mid C \mid 42 \mid \ldots$	defined variable, data constructor atomic literal, ...

- pointers instead of edges

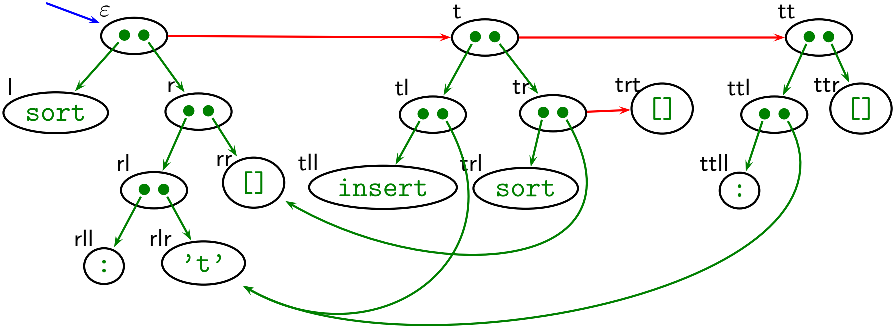
The Node Naming Scheme



Aim

- not distinguish isomorphic graphs
- avoid inconvenience of isomorphism classes

The Node Naming Scheme



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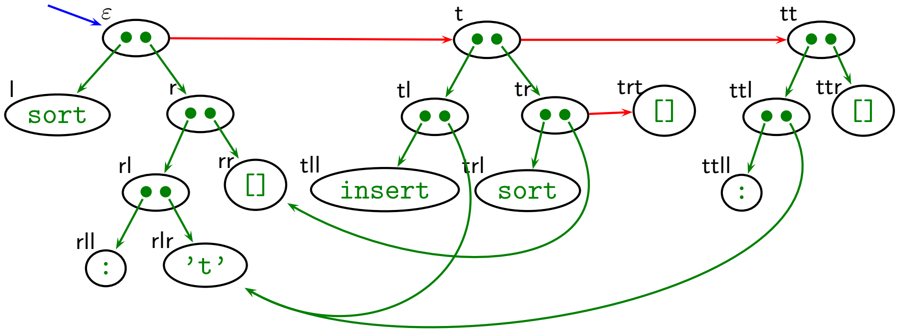
Solution

- standard representation with node describing path from root
- path at creation time (sharing later)
- path independent of evaluation order

The Node Naming Scheme II



- Reduction edge implicitly given through existence of node.
- Node encodes parent = top node of redex causing its creation:

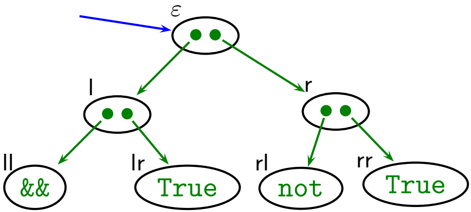
$$\begin{aligned} \text{parent}(n\text{t}) &= n \\ \text{parent}(n\text{l}) &= \text{parent}(n) \\ \text{parent}(n\text{r}) &= \text{parent}(n) \\ \text{parent}(\varepsilon) &= \text{undefined} \end{aligned}$$

- Easy to identify right-hand-side of rule: same parent.

Projections

- Reduction edge implicitly given through existence of node.
- Every redex should be parent of at least one node.
  (otherwise reduction unreachable from computation result)

```
True && x = x
not True = False
```



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⇒ A projection requires an indirection as result.

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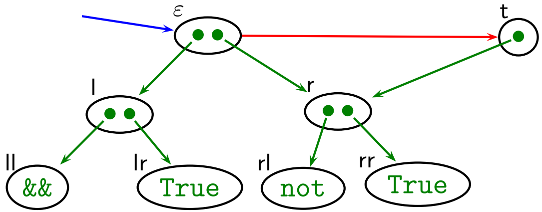

term constructor	T	$:=$	a	atom
		$\mid$	$n\ m$	application of nodes
		$\mid$	n	indirection
atom	a	$:=$	$x \mid C \mid 42 \mid \ldots$	variable, data constructor, ...

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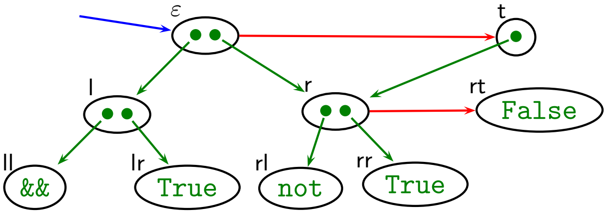

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The Trace: The Augmented Redex Trail (ART)

A trace G for initial term M and program P is a partial function from nodes to term constructors, $G : n \mapsto T$, defined by

- The unshared graph representation of M, $\mathrm{graph}_G(\varepsilon, M)$, is a trace.
- If G is a trace and
  - $L = R$ an equation of the program P,
  - σ a substitution replacing argument variables by nodes,
  - $\mathrm{match}_G(n, L\sigma)$,
  - $n\mathrm{t} \notin \mathrm{dom}(G)$,

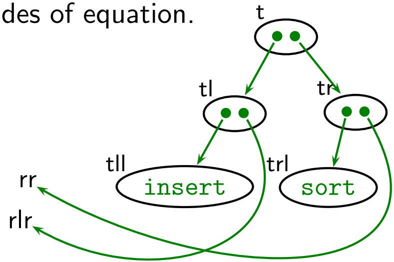
  then $G \cup \mathrm{graph}_G(n\mathrm{t}, R\sigma)$ is a trace.

No evaluation order is fixed.

Unshared Graph Representation

For the initial term and right-hand-sides of equation.

$$\text{graph}(\text{t}, \texttt{insert}\ \text{rlr}\ (\text{sort}\ \text{rr})) =$$



Definition

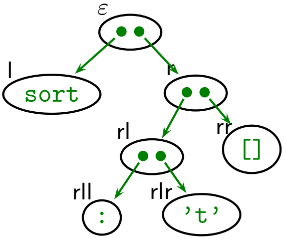
$$\text{graph}(n, a) = \{(n, a)\}$$

$$\text{graph}(n, m) = \{(n, m)\}$$

$$\text{graph}(n, M\,N) = \begin{cases} \{(n, M\,N)\} & \text{, if } M, N \text{ are nodes} \\ \{(n, M\,n\text{r})\} \cup \text{graph}(n\text{r}, N) & \text{, if only } M \text{ is a node} \\ \{(n, n\text{l}\,N)\} \cup \text{graph}(n\text{l}, M) & \text{, if only } N \text{ is a node} \\ \{(n, n\text{l}\,n\text{r})\} \cup \text{graph}(n\text{l}, M) \cup \text{graph}(n\text{r}, N) & \text{, otherwise} \end{cases}$$

Matching

Matching a node with an instance of the left-hand-side of an equation.



$\mathsf{match}_G(\varepsilon, \texttt{sort}\ (\mathsf{rlr}\texttt{:}\mathsf{rr}))$

Definition

$$\mathsf{match}_G(n, M) = \text{if } M \text{ is a node then } n = M \text{ else } \mathsf{matchT}_G(\mathsf{last}_G(n), M)$$

$$\mathsf{matchT}_G(a, M) = (a = M)$$

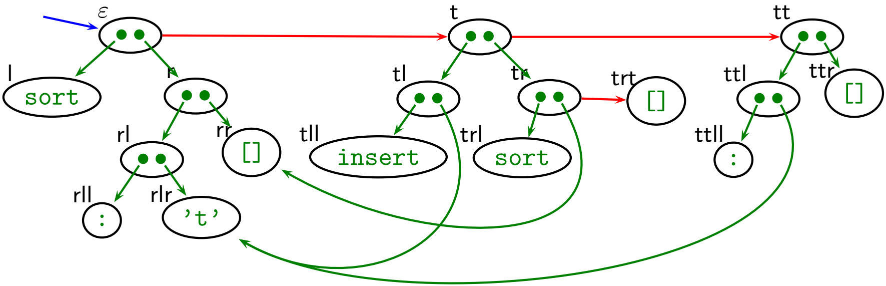
$$\mathsf{matchT}_G(n, M) = \mathsf{matchT}_G(\mathsf{last}_G(n), M)$$

$$\mathsf{matchT}_G(n\, o, M) = \exists N, O.\, (N\, O = M) \wedge \mathsf{match}_G(n, N) \wedge \mathsf{match}_G(o, O)$$

$$\mathsf{last}_G(n) = \text{if } nt \in \mathsf{dom}(G) \text{ then } \mathsf{last}_G(nt) \text{ else } G(n)$$

The Most Evaluated Form of a Node

A node represents many terms, in particular a most evaluated one.



$\text{mef}_G(\text{tr}) = $ `[]`

$\text{mef}_G(\varepsilon) = $ `(:) 't' []`

Definition

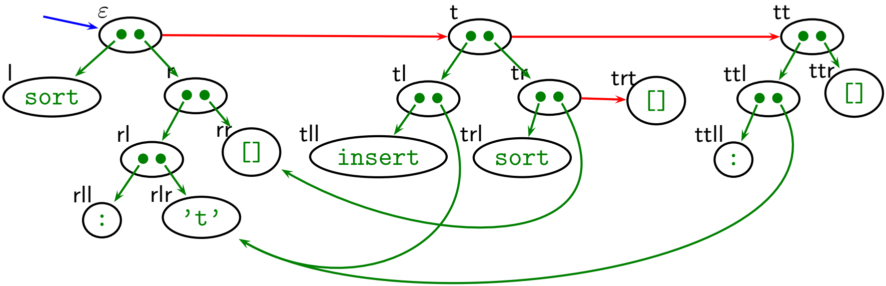
$$\text{mef}_G(n) = \text{mefT}_G(\text{last}_G(n))$$

$$\text{mefT}_G(a) = a$$

$$\text{mefT}_G(n) = \text{mef}_G(n)$$

$$\text{mefT}_G(n\,m) = \text{mef}_G(n)\,\text{mef}_G(m)$$

Redexes and Big-Step Reductions



$$\mathrm{redex}_G(\mathrm{t}) = \texttt{insert 't' []}$$

$$\mathrm{bigstep}_G(\mathrm{t}) = \texttt{insert 't' []} = \texttt{(:) 't' []}$$

Definition

For any redex node n, i.e., $nt \in \mathrm{dom}(G)$

$$\mathrm{redex}_G(n) = \begin{cases} \mathrm{mef}_G(m)\,\mathrm{mef}_G(o) & \text{, if } G(n) = m\,o \\ a & \text{, if } G(n) = a \end{cases}$$

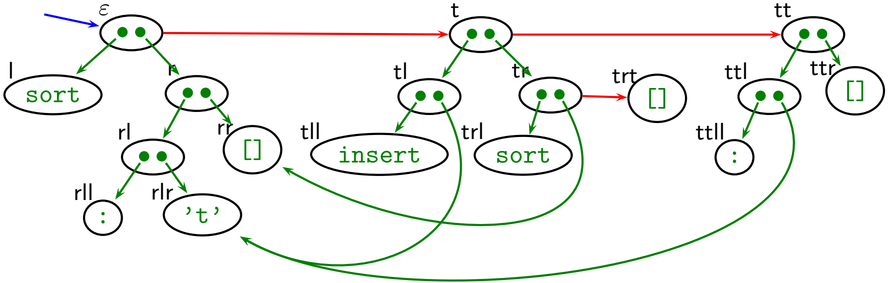
$$\mathrm{bigstep}_G(n) = \mathrm{redex}_G(n) = \mathrm{mef}_G(n)$$

Properties of the ART

- closed (no dangling nodes)
- domain prefix-closed
- acyclic
- strongly confluent
- no application contains a node ending in t
- only a node ending in t can be an indirection
- if $n\mathsf{l} \in \mathsf{dom}(G)$, then $G(n) = n\mathsf{l}\; m$
- if $n\mathsf{r} \in \mathsf{dom}(G)$, then $G(n) = m\; n\mathsf{r}$
- if $n\mathsf{t} \in \mathsf{dom}(G)$, then $\mathsf{redex}_G(n) = L\sigma$ and $\mathsf{reduct}_G(n) = R\sigma$ for some program equation $L = R$ and substitution σ

Give non-inductive definition of ART based on properties?

Reduct of a Small Step Reduction



$reduct_G(\varepsilon)$ = `insert 't' (sort [])`

Definition

$$\mathrm{reduct}_G(n) = \mathrm{reductB}_G(n\mathrm{t})$$

$$\mathrm{reductB}_G(n) = \begin{cases} a & \text{, if } G(n) = a \\ \mathrm{mef}_G(m) & \text{, if } G(n) = m \\ \mathrm{reductB}_G(n\mathrm{l})\,\mathrm{reductB}_G(n\mathrm{r}) & \text{, if } G(n) = n\mathrm{l}\; n\mathrm{r} \\ \mathrm{reductB}_G(n\mathrm{l})\,\mathrm{mef}_G(o) & \text{, if } G(n) = n\mathrm{l}\; o \text{ and } o \neq n\mathrm{r} \\ \mathrm{mef}_G(m)\,\mathrm{reductB}_G(n\mathrm{r}) & \text{, if } G(n) = m\; n\mathrm{r} \text{ and } m \neq n\mathrm{l} \\ \mathrm{mef}_G(m)\,\mathrm{mef}_G(o) & \text{, if } G(n) = m\; o,\ m \neq n\mathrm{l} \text{ and } o \neq n\mathrm{r} \end{cases}$$

Views of the Trace

- Observation of Expressions and Functions
- Following Redex Trails
- Algorithmic Debugging

Observation of Expressions and Functions

Observation of Expressions and Functions

Observation of function sort:

```
sort "sort" = "os"
sort "ort" = "o"
sort "rt" = "r"
sort "t" = "t"
sort "" = ""
```

Observation of function insert:

```
insert 's' "o" = "os"
insert 's' "" = "s"
insert 'o' "r" = "o"
insert 'r' "t" = "r"
insert 't' "" = "t"
```

Big step reductions of redex nodes.

Following Redex Trails

Following Redex Trails

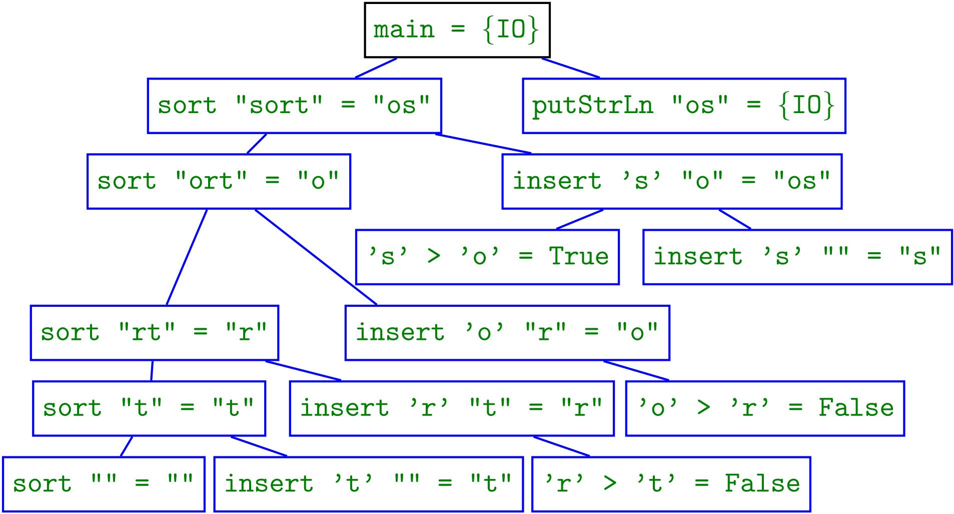
```
Output: ------------------------------------------------
os\n

Trail: ------- Insert.hs line: 10 col: 25 ------------------
<- putStrLn "os"
<- insert 's' "o" | if True
<- insert 'o' "r" | if False
<- insert 'r' "t" | if False
<- insert 't' []
<- sort []
```

- Go backwards from observed failure to fault.
- Which redex created this expression?
- To prove: every reduction step reachable from final result.

Algorithmic Debugging

```
sort "sort" = "os"?   n

insert 's' "o" = "os"?   y

sort "ort" = "o"?   n

insert 'o' "r" = "o"?   n

Bug identified:
  "Insert.hs":8-9:
  insert x [] = [x]
  insert x (y:ys) = if x > y then y:(insert x ys) else x:ys
```

The Evaluation Dependency Tree



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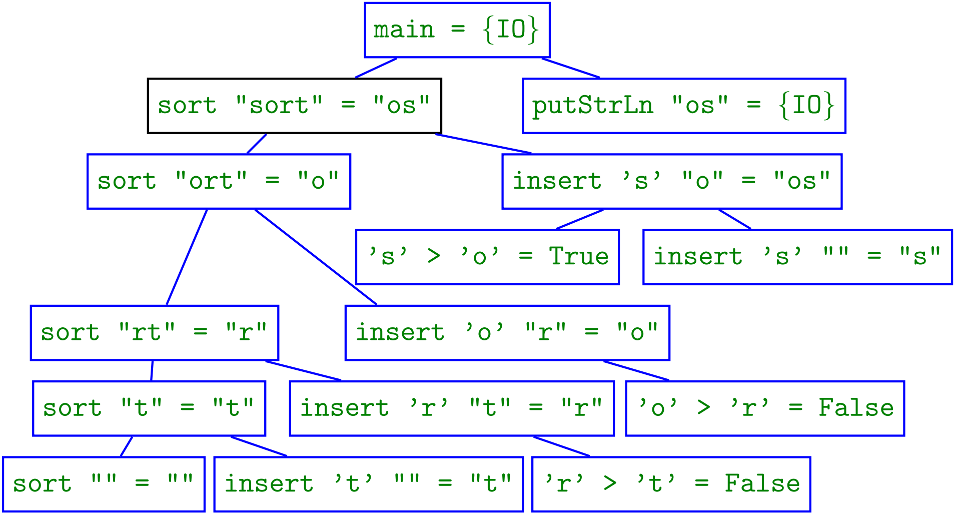

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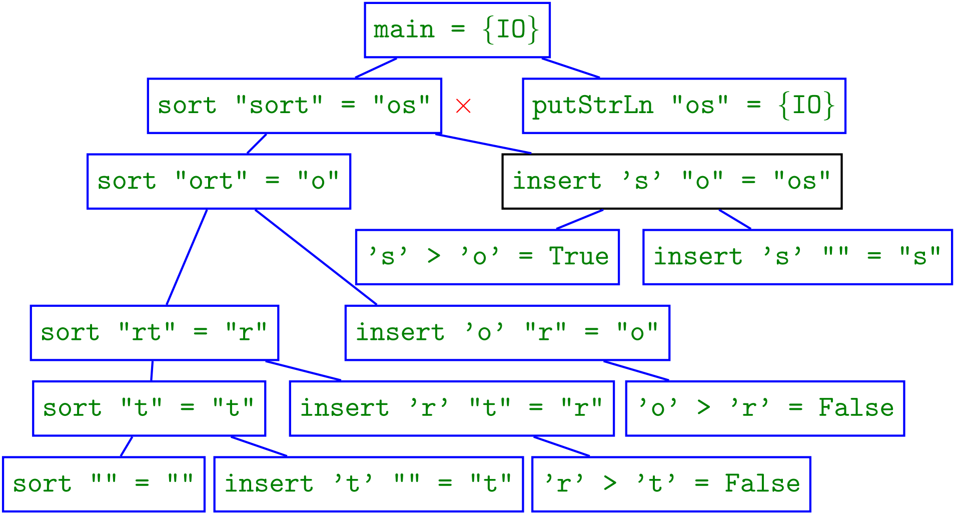

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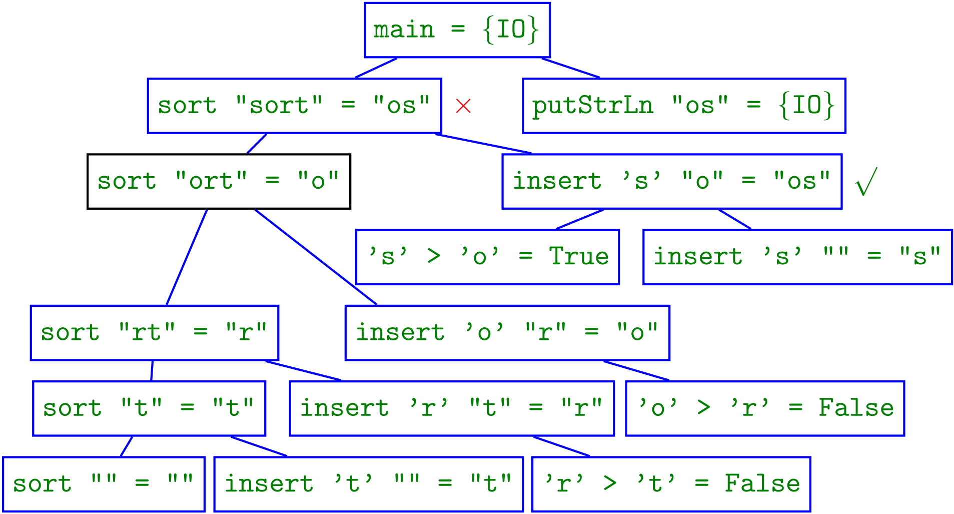

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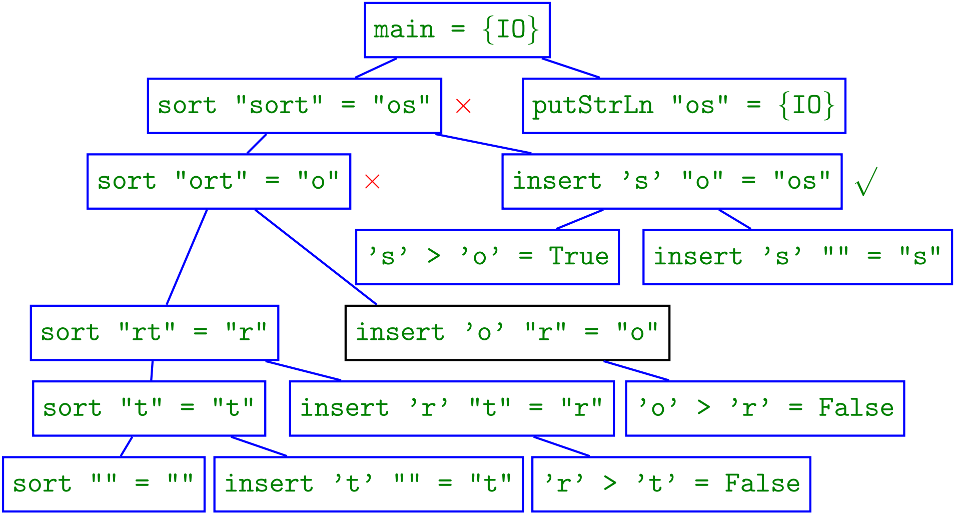

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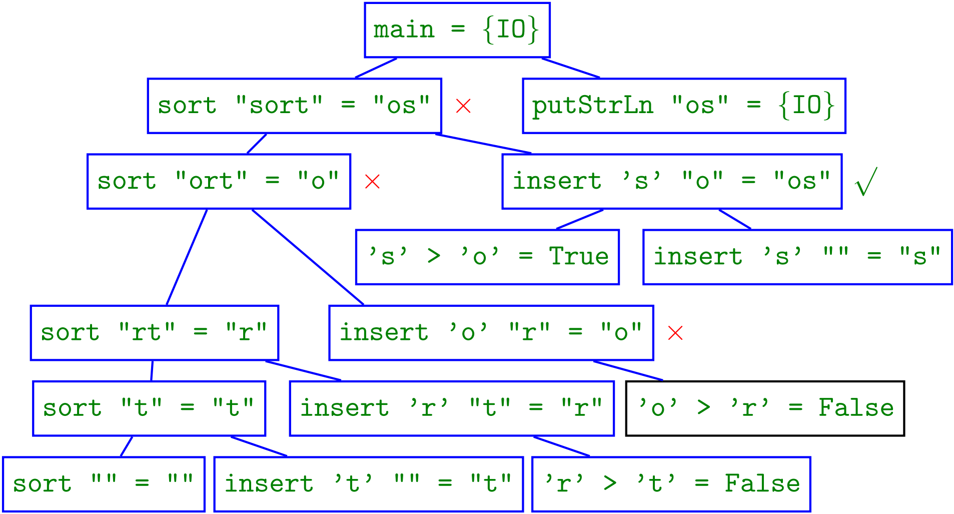
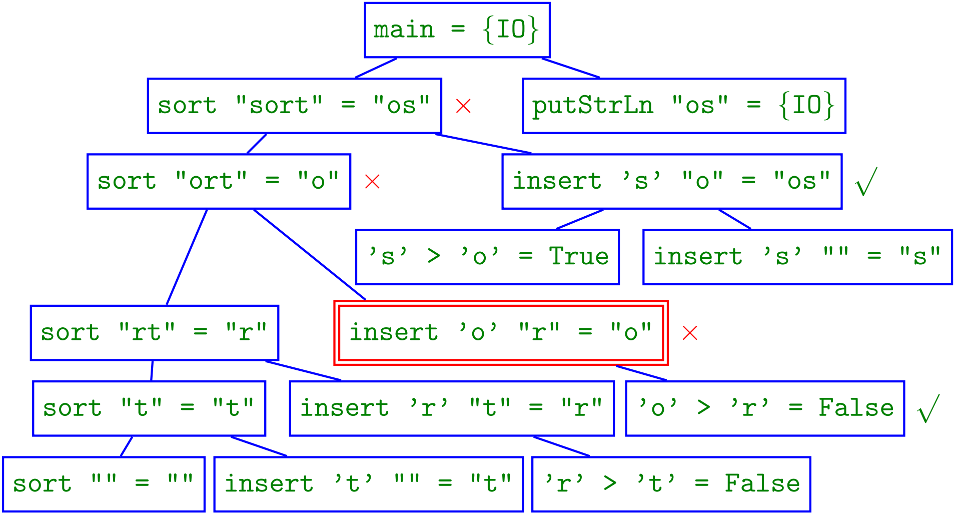
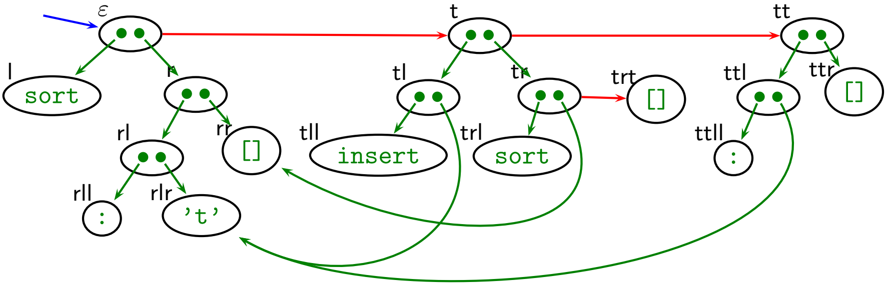
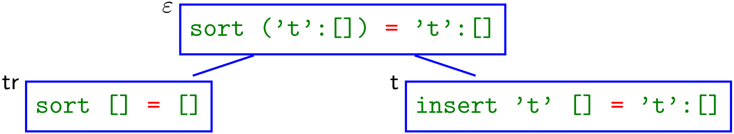


The Evaluation Dependency Tree



The ART and the Evaluation Dependency Tree

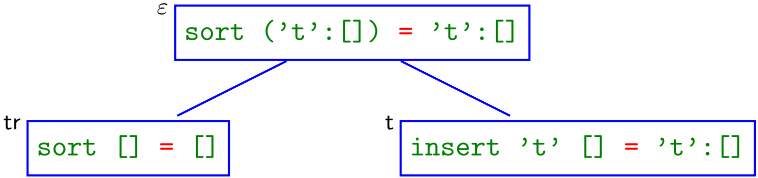




- Every redex node n yields a tree node n labelled $\text{bigstep}_G(n)$.
- Tree node n is child of tree node $\text{parent}(n)$.
- Usually root label $\text{bigstep}_G(\varepsilon) = \texttt{main = ...}$

Correctness of Algorithmic Debugging: The Property

If node n incorrect and all its children correct, then node n faulty, i.e., its equation is faulty.



Definition

Tree node n incorrect $\Leftrightarrow$ $\mathrm{redex}_G(n) \not\cong_I \mathrm{mef}_G(n)$.
Tree node n faulty $\Leftrightarrow$ $\mathrm{redex}_G(n) \not\cong_I \mathrm{reduct}_G(n)$.

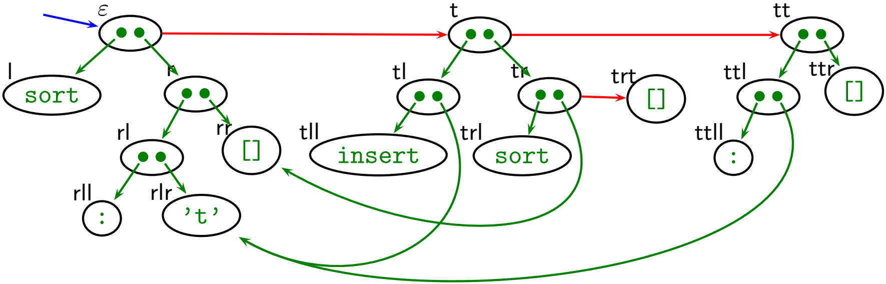
If tree node n faulty, then for its program equation $L = R$ exists substitution σ such that $L\sigma \not\cong_I R\sigma$.

Correctness of Algorithmic Debugging: Main Theorem

Theorem

Let n be a redex node. If for all redex nodes m with $parent(m) = n$ we have $redex_G(m) \cong_I mef_G(m)$, then $reduct_G(n) \cong_I mef_G(n)$.

With $\mathrm{redex}_G(n) \not\cong_I \mathrm{mef}_G(n)$ follows $\mathrm{redex}_G(n) \not\cong_I \mathrm{reduct}_G(n)$.

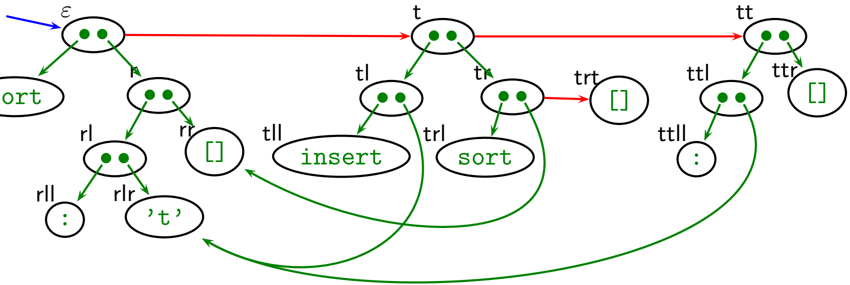


Correctness of Algorithmic Debugging: Proof

Proof.

Generalise property: Let $n \in \mathrm{dom}(G)$. If for all redex nodes m with $\mathrm{parent}(m) = \mathrm{parent}(n)$ we have $\mathrm{redex}_G(m) \cong_\mathrm{I} \mathrm{mef}_G(m)$, then $\mathrm{reductB}_G(n) \cong_\mathrm{I} \mathrm{mef}_G(n)$.

Induction over $\mathrm{hight}_G(n) = \max\{|o| \mid o \in \{\mathrm{l}, \mathrm{r}\}^* \wedge no \in \mathrm{dom}(G)\}$. $\square$



Future Work

- Still play with definitions.
- Extend model further:
  - Drop non-needed nodes from ART (unevaluated expressions).
  - Model run-time error with error value.
  - Allow local function definitions ($\Rightarrow$ free variables).
  - Share reductions of constants ($\Rightarrow$ cycles in graph).
  - Describe strict and mixed semantics.
- Prove further properties.

Conclusions

- Simple model amenable to proof.
- Contains a wealth of information about computation.
- Models real-world trace of Haskell tracer Hat.
- Proved correctness of algorithmic debugging.