Expressiveness of Spider Diagrams

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12 February 2007

Overview

- Spider diagrams: examples, what and why.
- Logical questions: reasoning and expressiveness.
- Equivalences, and the role of negation.
- Monadic first-order logic.
- SD to MLE.
- Examining models of MLE.
- Model classes as spider diagrams.
- MLE to SD
- Summing up.

Spider diagrams

- Describe sets, their membership and interrelations.
- The absence of a zone means that it is empty: e.g., a\(\cap\)c.
- Shading means that a zone contains no more elements than indicated (upper limit).
- A spider represents an element at one of its feet.
Spider diagrams

- a contains at least one element.
- a-b contains at most one element.
- a∩c, b∩c are empty.

Unitary spider diagrams

- Every unitary spider diagram is satisfiable. For example:
  - a = \{1,2,3\}, b = \{2,3\}, c = \{4,5\}.
- Minimal models can be read off from the diagram.
  - a = b = \{1\}, c = \{}.
  - a = \{1\}, b = c = \{}.

General spider diagrams

- Combine unitary diagrams with \&, \lor (and \neg).
General spider diagrams

- Not all spider diagrams are satisfiable.

Why spider diagrams?

- Improvement on Venn diagrams, which quickly become unreadable because all 2^n zones need to be shown.
- Useful in visual description of set-theoretic constraints, such as those that apply in OO modelling.
- Very pleasant logical properties.
- Intuitive.
- A case study for more complicated visual representations, of constructs like \exists and \forall.

Logical questions

- Which diagrams are satisfiable (i.e. true in some model)?
- Which diagrams are valid (i.e. true in all models)?
- Is there a notion of logical equivalence between pairs of diagrams?
- Is there a notion of logical consequence?
- What properties do equivalence and consequence have?
- What can be expressed using spider diagrams?
Logical equivalences

- Splitting spiders: remove the disjunction implicit in a spider, and turn it into an explicit disjunction.
- Adding a contour: a contour can be added, with various adjustments.
- A shaded empty zone can be added to or removed from the diagram.
- etc ...

Splitting spiders

- Splitting spiders: remove the disjunction implicit in a spider, and turn it into an explicit disjunction.

Adding a contour

- A contour can be added, with all appropriate intersections.
- Spider feet need to be replicated over any new zones created.
Shaded empty zones

- A non-existent zone can be replaced by a shaded empty zone, and vice versa.

Why not $\neg$?

- Negation is a derived operator.
- $a \setminus b$ contains exactly one element.
- negation: $a \setminus b$ contains 0, or 2+ ...

Why not $\neg$?

- For a general spider diagram, split all spiders into disjunctions of $\alpha$-diagrams, where each spider has one foot.
- Then use de Morgan's laws repeatedly to push all negations inwards to the $\alpha$-diagrams, and negate as above.
Expressiveness

- What can be expressed by means of spider diagrams?
- Can we find a more traditional, non-visual, logical system which has the same expressive power?
- Relate systems by translation from one system to another…
- … or by showing how the models of the formulas of the two systems are related.

Monadic logic (MLE)

- A monad is a single unit.
- Monadic logic: one-place predicates only.
- Atomic formulas: $P(x), x=y$
- Propositional combinations: $F \land G, F \lor G, F \Rightarrow G, \neg F$
- Quantifications: $\exists x.F, \forall x.F$

Spider diagrams to MLE

- Each contour is a predicate.
- Enough to translate $\alpha$-diagrams … the rest is logical combination.
- Missing zones: state that there’s no element.
- Spiders: distinct spiders are distinct elements; not the usual rule for $\exists$.
- Shaded areas: express the upper bound here using the $\forall$ quantifier.
Example: SD to MLE

- $\exists x \exists y. (A(x) \land \neg B(x) \land A(y) \land \neg B(y) \land x \neq y)$
- $\exists x. (A(x) \land B(x) \land \forall y. ((A(y) \land B(y)) \Rightarrow x = y)$
- $\neg \exists x. (\neg A(x) \land B(x))$

SD = MLE?

- The translation shows that spider diagrams are no more expressive than MLE, so SD $\subseteq$ MLE.
- Another route: look at the sets of models for MLE formulas and for spider diagrams.

MLE $\subseteq$ SD?

- One mechanism is to give a translation from MLE to SD ... no obvious way of doing this.

MLE: first key insight

- N nested quantifiers introduce N names.
- A sentence of the form $Qx_1 Qx_2 \ldots Qx_N F$ can talk about at most N individuals: $x_1, x_2, \ldots, x_N$.
- Using ‘$=’ these $x_i, x_2, \ldots, x_N$ can be made distinct.
- Examples:
  - $\exists x. \exists y. (A(x) \land \neg B(x) \land A(y) \land \neg B(y) \land x \neq y)$
  - $\exists x. (A(x) \land B(x) \land \forall y. ((A(y) \land B(y)) \Rightarrow x = y)$
  - $\neg \exists x. (\neg A(x) \land B(x))$
MLE: second key insight

- A structure $M$ for a sentence using $P_1, \ldots, P_k$ is characterised by the sets $M(X) = \cap_{i \in X} P_i \cap \cup_{i \notin X} \neg P_i$
where $X$ ranges over subsets of $\{1, 2, \ldots, k\}$.
- $X$ ranges over all the possible combinations of $P_i / \neg P_i$.

Hypothesis

- MLE formulas involving $P_i, \ldots, P_k$ have the effect of describing each of the $2^k$ regions given by a choice of $X \subseteq \{1, \ldots, k\}$.
- MLE formulas involving quantifier nesting $N$ can control the size of $M(X)$ to be $0, 1, 2, \ldots, N-1, \geq N$.
- Models of MLE formulas are either small or large, i.e. larger than $N.2^k$.
- If large, then can add elements to all large regions and remain a model.

Definition

Two structures $M_1$ and $M_2$ are similar with respect to the sentence $S$, with quantifier nesting $N$ and using predicates $P_1, P_2, \ldots, P_k$, iff

for all $X \in P(\{1, 2, \ldots, k\})$, $M_1(X) = M_2(X)$ or $|M_1(X) \cap M_2(X)| \geq N$

for all $X, Y \in P(\{1, 2, \ldots, k\}), X \neq Y$
$M_1(X) \cap M_2(Y) = \emptyset$
Result

If $M_1$ and $M_2$ are similar with respect to the sentence $S$, then for any subformula $G$ of $S$ and assignment to the free variables of $G$ of values in $U_1 \cap U_2$,

$M_1$ models $G$ if and only if $M_2$ models $G$.


Proof sketch

The crucial case: when $G$ has the form $\exists x. H$.

There's witness $a$ for $H$ lying in $M_1(X)$.
If $M_1(X)$ is small; $a$ is also a witness in $M_2$.
If $M_1(X)$ is large, and $a$ in $U_2$, we're done.
If $a$ not in $U_2$, can pick an unmentioned $b$ in $M_2(X)$, and argue that $b$ is itself a witness in $M_2$, by automorphism argument.

Spider diagram models

This zone has upper and lower limits on its size.

This zone has no limits on its size.

This zone has a lower limit on its size.
Models of MLE formulas

- Models of MLE formulas are either small or large, i.e. larger than N.2^k.
- If large, then can add elements to all large regions and remain a model.

MLE ⊆ SD

- Represent each small model as a fully shaded unitary SD.
- Represent each (minimal) large model as a unitary SD with shading in small regions.
- Disjunction of these gives the representing spider diagram.

Constructive?

- Yes: run through all the potential small and minimal large models, checking whether they are indeed models.
- No: not in the sense that there's a direct, syntactic algorithm going from MLE sentences to spider diagrams.
Refinements

- Can make the representing SD substantially smaller by applying spider-creating rules, and coalescing small and large models when possible.

Commentary

- Like a quantifier elimination procedure, but not quite the same.
- QE procedure can be used directly on ML formulas; doesn’t seem to generalise to MLE.

Conclusion

- Spider diagrams give an attractive, intuitive representation of properties of sets and their elements.
- Spider diagrams are obviously examples of sentences in monadic first-order logic with equality.
- Less obviously, all MLE sentences can be represented by spider diagrams.
- The proof of this goes via an analysis of the class of models of MLE sentences, and thus to a set of SDs characterising the model class.