Consistency in Rewriting

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Personal Stuff

- ...in the late 1980s...
- ...in Edinburgh in the 1990s...

In recent years...

...I have been occupied with a long-standing open consistency problem in TRSs:

Do non-ω-overlapping TRS have unique nomal forms?

Consistency? It is equivalent to:

Do they have a consistent equational theory?

Not as esoteric as it sounds...

- TRS are known to be confluent (and hence consistent) if they are LL and non-overlapping [Rosen 1973]
- without LL confluence (and UN) is lost
- and so is consistency, but...
- but: if we strengthen the non-overlapping condition...

Example

$F(x,x) \to A$

$F(x,C(x)) \to B$

$E \to C(E)$

is non-UN, can be made inconsistent, but has ω-overlaps

Term Rerwriters...

- trigger warning: some sweeping generalisations coming up
- Term Rewriters do not like...
  - semantics, models, category theory
- they do like...
  - syntax, formal proofs (semi-automatic)

Step 1: reduce the problem

- for every TRS there is a constructor-TRS with an "equivalent" equational theory
- construction: double the signature, use signature morphisms, turn patterns into constructor terms
- note: no s.m. in Terese
- note: for non-ω-overlapping TRSs the result is near identical

Step 2: find a consistent invariant

- confluent TRSs: joinability $\downarrow$ is a consistent invariant
- we need similar closure principles as joinability for $\Downarrow$, but in addition a stronger one for operating on subterms
- $t \Downarrow u \wedge u \overline{\Downarrow} s \Longrightarrow t \Downarrow s$ (weak transitivity)
- informally: the relation $\overline{\Downarrow}$ means that terms share root symbol, and their subterms are related by $\Downarrow$

Step 3: prove the invariant

- show that all TRS-equivalent terms are related by $\Downarrow$
- how?
- use finite coalgebras of the signature functor
- (very similar to: finite set of terms represented in memory)

What?

- the overall argument is:
- we have consistency, because there is no smallest counterexample
- take the smallest set of terms you need to prove t=u but you cannot prove $t \Downarrow u$
- assume for all strict sub-coalgebras there is no counterexample
- reach a contradiction

In particular...

- Represent proofs of $t \Downarrow u$ in a coalgebra A as union/find structures
- these are extendible and so...
- ...one such structure acts as the universal proof of $\Downarrow$ in A, i.e. any two terms of A related by $\Downarrow$ are linked by the structure
- because union/find structures represent equivalences, $\Downarrow$ is an equivalence