Personal Stuff

• ...in the late 1980s...
• ...in Edinburgh in the 1990s...
In recent years...

...I have been occupied with a long-standing open consistency problem in TRSs:

Do non-\(\omega\)-overlapping TRS have unique normal forms?

Consistency? It is equivalent to:

Do they have a consistent equational theory?
Not as esoteric as it sounds...

- TRS are known to be confluent (and hence consistent) if they are LL and non-overlapping [Rosen 1973]
- without LL confluence (and UN) is lost
- and so is consistency, but...
- but: if we strengthen the non-overlapping condition...
Example

F(x,x) → A
F(x,C(x)) → B
E → C(E)

is non-UN, can be made inconsistent, but has $\omega$-overlaps
Term Rerwriters...

- trigger warning: some sweeping generalisations coming up
- Term Rerwriters do not like...
  - semantics, models, category theory
- they do like...
  - syntax, formal proofs (semi-automatic)
Step 1: reduce the problem

- for every TRS there is a constructor-TRS with an "equivalent" equational theory
- construction: double the signature, use signature morphisms, turn patterns into constructor terms
- note: no s.m. in Terese
- note: for non-$\omega$-overlapping TRSs the result is near identical
Step 2: find a **consistent** invariant

- confluent TRSs: joinability $\downarrow$ is a consistent invariant
- we need similar closure principles as joinability for $\downarrow$, but in addition a stronger one for operating on subterms
- $t \downarrow u \land u \downarrow s \Rightarrow t \downarrow s$ (weak transitivity)
- informally: the relation $\downarrow$ means that terms share root symbol, and their subterms are related by $\downarrow$
Step 3: prove the invariant

- show that all TRS-equivalent terms are related by $\Downarrow$
- how?
- use finite coalgebras of the signature functor
- (very similar to: finite set of terms represented in memory)
What?

• the overall argument is:
• we have consistency, because there is no smallest counterexample
• take the smallest set of terms you need to prove $t = u$ but you cannot prove $t \not\geq u$
• assume for all strict sub-coalgebras there is no counterexample
• reach a contradiction
In particular...

- Represent proofs of $t \downarrow u$ in a coalgebra $A$ as **union/find** structures.
- These are extendible and so...
- ...one such structure acts as the universal proof of $\downarrow$ in $A$, i.e. any two terms of $A$ related by $\downarrow$ are linked by the structure.
- Because union/find structures represent equivalences, $\downarrow$ is an equivalence.