Orthogonality and Weak Convergence in Infinitary Rewriting

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What you probably know

- in orthogonal iTRSs weak/strong reduction fails to be confluent in the presence of collapsing rules (for metric d_∞)
- $F(x) \to x, G(x) \to x$ is a counterexample
- in the absence of collapsing rules strong reduction is confluent for orthogonal iTRSs
- what about weak reduction?

What some of you may know

- even without collapsing rules, weak reduction may not be confluent
- Simonsen's counterexample from 2004:

$$F\left(G^k(C), x, y\right) \to F\left(G^{k+1}(C), y, y\right) \text{ for even k}$$

$$F\left(G^k(C), x, y\right) \to F\left(G^{k+1}(C), A, y\right) \text{ for odd k}$$

$$A \to B$$

- we have: $F(C, A, A) \twoheadrightarrow_w F(G^\infty, A, A)$ and $F(C, A, A) \to F(C, A, B)$ but no common weak reduct

Issues with Simonsen's example

- ...as listed on open problems in rewriting:
  - system is infinite
  - depth of lhs is unbounded
  - not right-linear
  - right-hand sides not in normal form

The latter are really minor points, except the infiniteness issue: with infinitely many rules we can make iTRSs behave non-continuously.

Another issue

- didn't I tell you this morning that $\twoheadrightarrow_w$ is pants anyway, because it's (in general) not closed under its own construction principle?
- Simonsen's example is a case in point: the adherence relation $\twoheadrightarrow_a$ (the one we get from the fixpoint construction) is confluent here, as we have $F(C, A, B) \twoheadrightarrow_a \ F(G^\infty, A, B)$

Explanation

- for even k:
  $F\left(G^{k}(C), A, B\right) \to F\left(G^{k+1}(C), B, B\right) \to F\left(G^{k+2}(C), A, B\right)$
- hence $F\left(G^{k}(C), A, B\right) \twoheadrightarrow_{a} F\left(G^{k+2}(C), A, B\right)$ because $\twoheadrightarrow_{a}$ contains $\to$ and is transitive
- hence $F(C, A, B) \twoheadrightarrow_{a} F(G^{\infty}, A, B)$ because it is closed under weak limits

What you did not care about

- is adherence confluent for orthogonal, non-collapsing iTRS?
- No!
- And we do not need infinitely many rules to show this either...

Counterexample

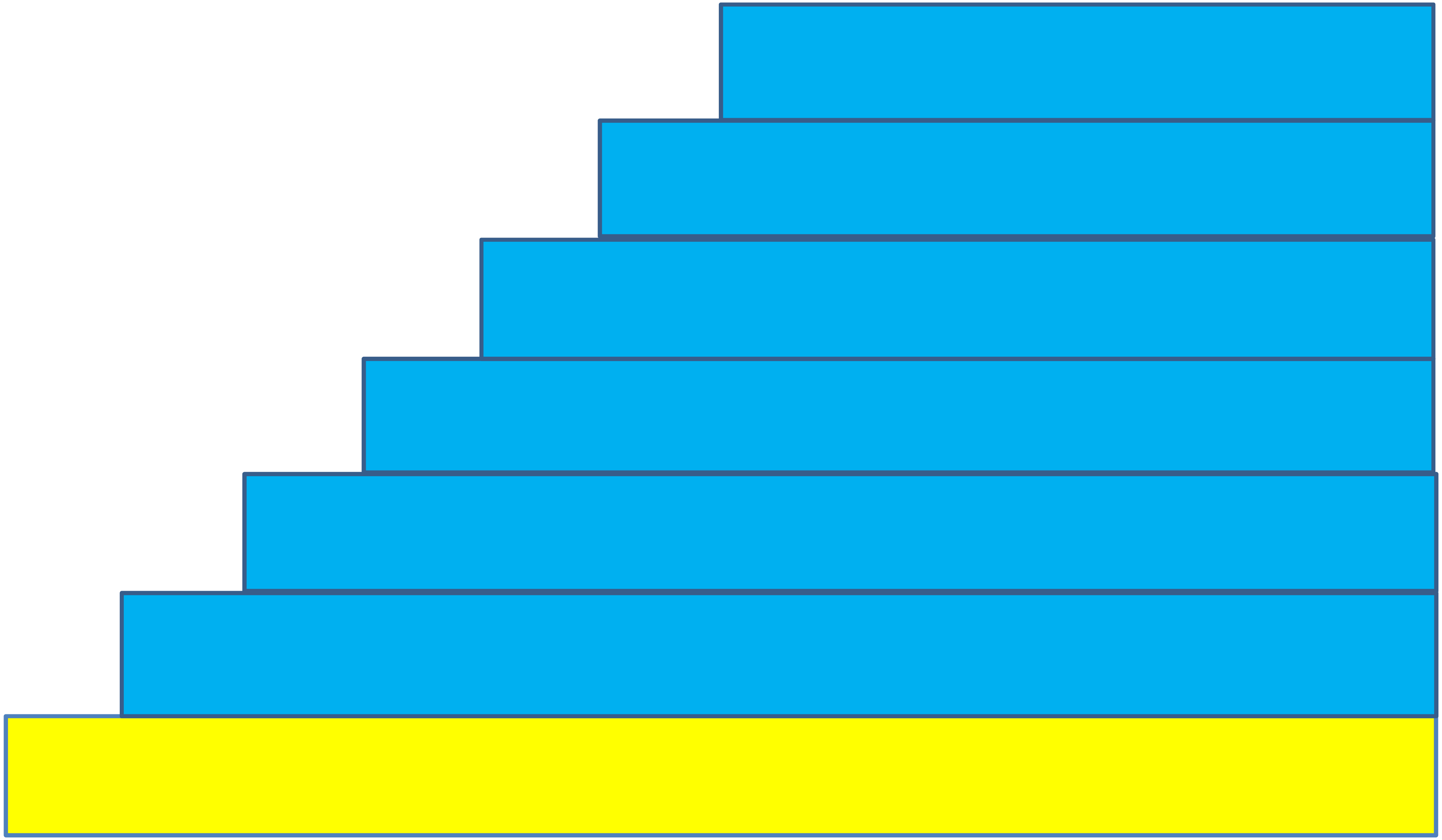
$$T\big(C(x, y)\big) \to T(y)$$
$$S(A) \to A$$

The second rule is not actually necessary, it is just used here for presentational purposes.

Think of C as a list-cons.

Imagine we have an infinite list, containing the values A, S(A), S(S(A)), etc. (in that order)

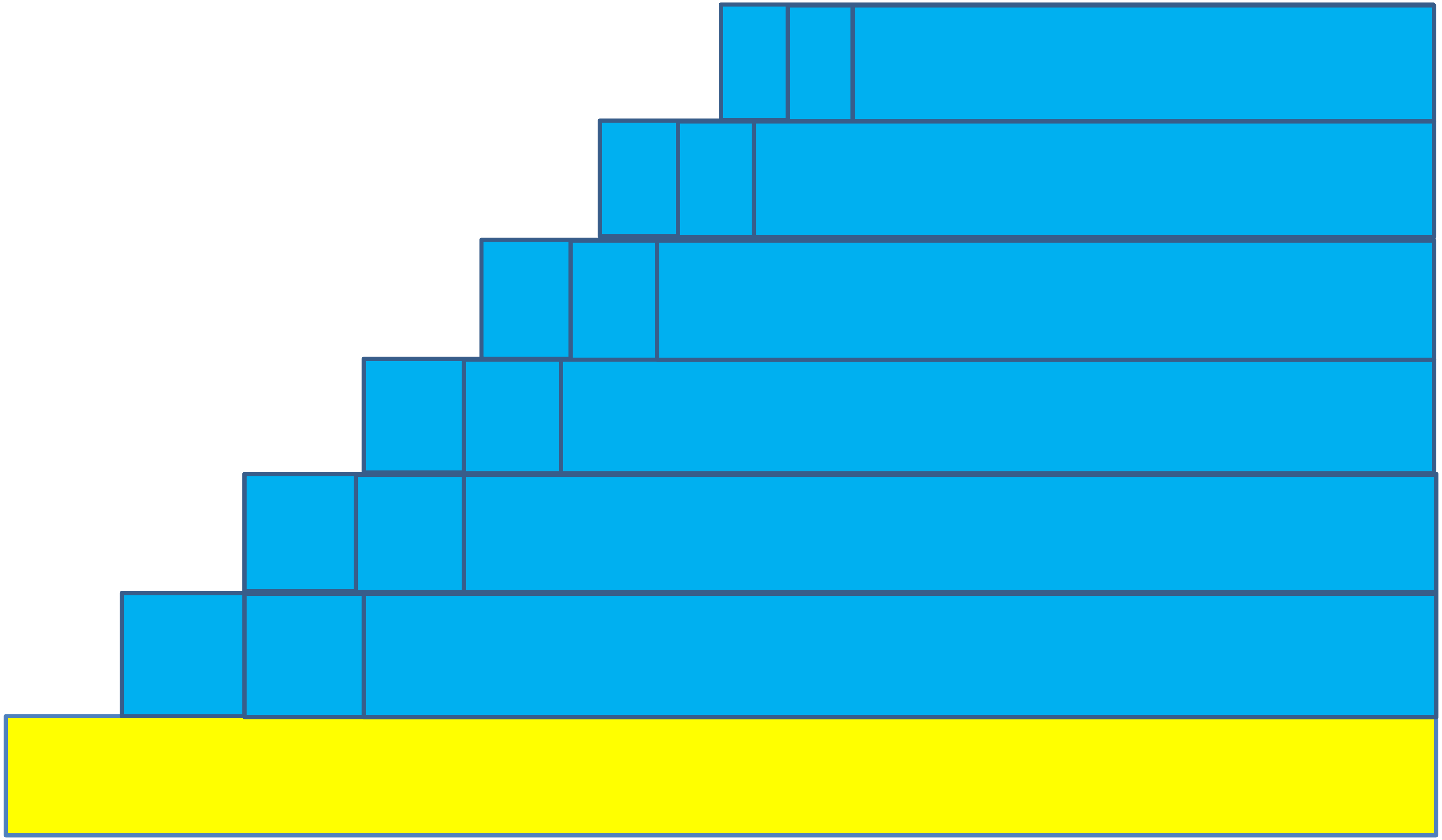
Visualisation



Explanation

- think of the yellow bar as all the A symbols in the infinite list
- the blue stuff on top of it as all the S symbols applied to the A symbols, so none for the first
- there are two things we can do to this:
  - make a complete development of all S-redexes (equivalent to sticking an A in front of the list)
  - apply the T-rule repeatedly (removing the first element)

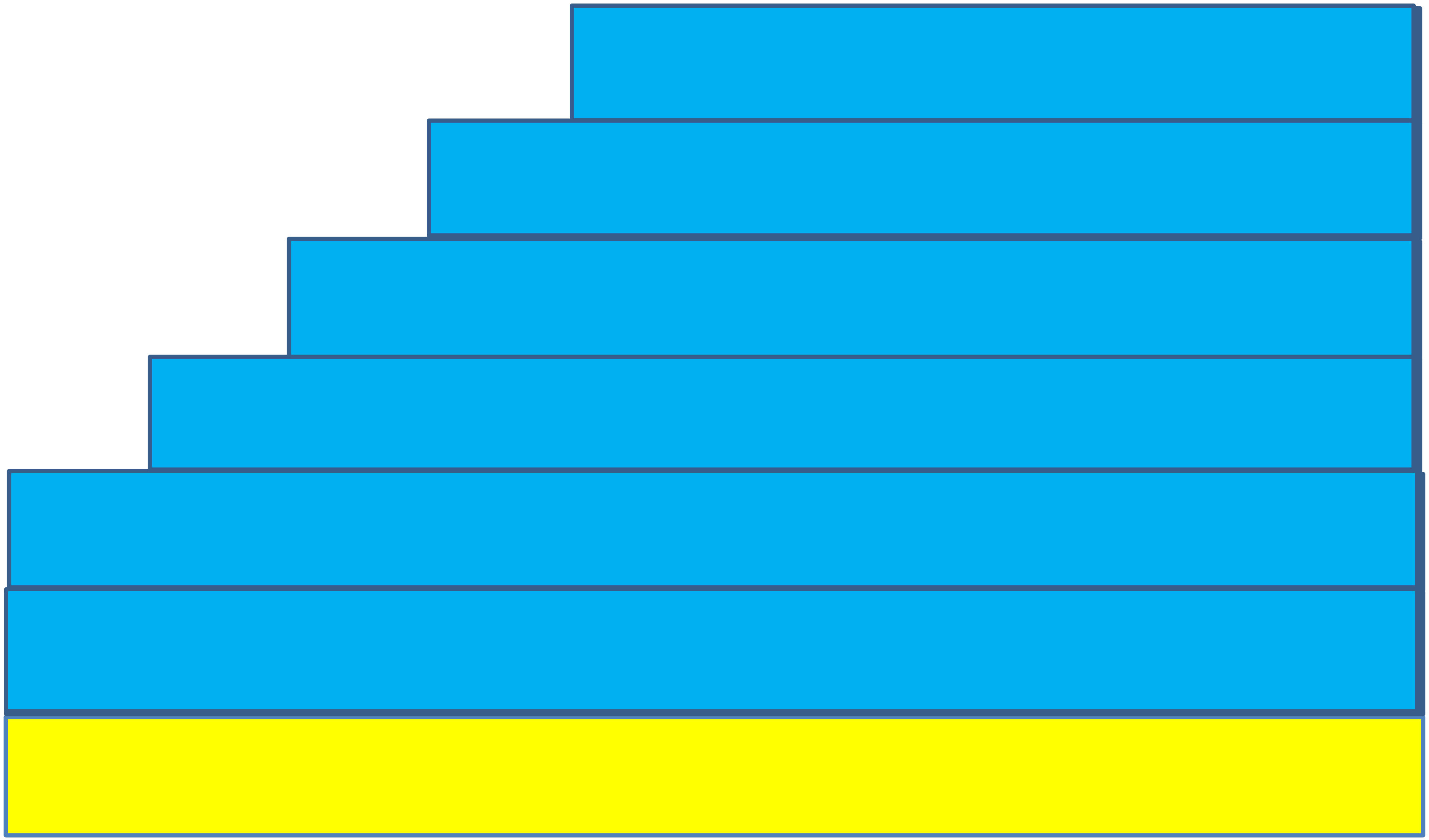
Visualisation, case 1



In the limit

- just an infinite list of As (a big yellow line)

Visualisation, case 2



In the limit

- just an infinite list of infinitely built-up Ss

And

- both these limits only reduce to themselves
- hence neither $\twoheadrightarrow_w$ nor $\twoheadrightarrow_a$ nor even $\twoheadrightarrow_p$ (transitive and pointwise closure) are confluent for the example
- note: we could easily add rules orthogonally to generate our infinite term, as opposed to start with it

Weirdly

- $\twoheadrightarrow_t$ (topological & transitive closure) actually is confluent for the example

Not only that – this relation is even confluent for our very first counterexample, with the collapsing rules!

Challenges

- Is $\twoheadrightarrow_t$ confluent for orthogonal iTRS, possibly all of them?
- Is $\twoheadrightarrow_w$ confluent for orthogonal & converging iTRSs? Note that for converging iTRSs the relations $\twoheadrightarrow_w$ and $\twoheadrightarrow_a$ coincide.
  - aside: iTRSs with two collapsing rules are not converging, so this could be quite general