Orthogonality and Weak Convergence in Infinitary Rewriting

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What you probably know

- In orthogonal iTRSs weak/strong reduction fails to be confluent in the presence of collapsing rules (for metric $d_\infty$)
- $F(x) \to x, G(x) \to x$ is a counterexample
- In the absence of collapsing rules strong reduction is confluent for orthogonal iTRSs
- What about weak reduction?
What some of you may know

• even without collapsing rules, weak reduction may not be confluent

• Simonsen’s counterexample from 2004:
  \[ F(G^k(C), x, y) \rightarrow F(G^{k+1}(C), y, y) \quad \text{for even } k \]
  \[ F(G^k(C), x, y) \rightarrow F(G^{k+1}(C), A, y) \quad \text{for odd } k \]
  \[ A \rightarrow B \]

• we have: \[ F(C, A, A) \rightarrow_w F(G^\infty, A, A) \]
  \[ F(C, A, A) \rightarrow F(C, A, B) \]
  but no common weak reduct
Issues with Simonsen’s example

• ...as listed on open problems in rewriting:
  – system is infinite
  – depth of lhs is unbounded
  – not right-linear
  – right-hand sides not in normal form

The latter are really minor points, except the infiniteness issue: with infinitely many rules we can make iTRSs behave non-continuously.
Another issue

• didn’t I tell you this morning that $\rightarrow_w$ is pants anyway, because it’s (in general) not closed under its own construction principle?

• Simonsen’s example is a case in point: the adherence relation $\rightarrow_a$ (the one we get from the fixpoint construction) is confluent here, as we have $F(C, A, B) \rightarrow_a F(G^\infty, A, B)$
Explanation

• for even k:
\[ F(G^k(C), A, B) \to F(G^{k+1}(C), B, B) \to F(G^{k+2}(C), A, B) \]

• hence \( F(G^k(C), A, B) \Rightarrow_a F(G^{k+2}(C), A, B) \) because \( \Rightarrow_a \) contains \( \to \) and is transitive

• hence \( F(C, A, B) \Rightarrow_a F(G^\infty, A, B) \) because it is closed under weak limits
What you did not care about

• is adherence confluent for orthogonal, non-collapsing iTRS?

• No!

• And we do not need infinitely many rules to show this either...
Counterexample

\[ T(C(x, y)) \to T(y) \]
\[ S(A) \to A \]

The second rule is not actually necessary, it is just used here for presentational purposes.

Think of \( C \) as a list-cons.

Imagine we have an infinite list, containing the values \( A, S(A), S(S(A)) \), etc. (in that order)
Explanation

• think of the yellow bar as all the A symbols in the infinite list
• the blue stuff on top of it as all the S symbols applied to the A symbols, so none for the first
• there are two things we can do to this:
  – make a complete development of all S-redexes (equivalent to sticking an A in front of the list)
  – apply the T-rule repeatedly (removing the first element)
Visualisation, case 1
In the limit

• just an infinite list of As (a big yellow line)
Visualisation, case 2
In the limit

• just an infinite list of infinitely built-up Ss
And

• both these limits only reduce to themselves
• hence neither →_w nor →_a nor even →_p
  (transitive and pointwise closure) are confluent for the example
• note: we could easily add rules orthogonally to generate our infinite term, as opposed to start with it
Weirdly

• $\Rightarrow_t$ (topological & transitive closure) actually is confluent for the example

Not only that – this relation is even confluent for our very first counterexample, with the collapsing rules!
Challenges

• Is \( \rightarrow_t \) confluent for orthogonal iTRS, possibly all of them?

• Is \( \rightarrow_w \) confluent for orthogonal & converging iTRSs? Note that for converging iTRSs the relations \( \rightarrow_w \) and \( \rightarrow_a \) coincide.
  – aside: iTRSs with two collapsing rules are not converging, so this could be quite general