# Normal Forms and Infinity

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# Motivation

- ...behind this work was not infinitary rewriting at all
- it was an investigation of a long-standing open problem from the world of finite rewriting
- we were merely using infinitary rewriting in the construction of a model
- ...and in this model, the standard normal forms of infinitary rewriting were not all “passive data”
- for now, we will ignore this starting point and start from the basics

# Normal Forms

a normal form w.r.t. a relation $R$ is...

...a term $t$ such that $\neg\exists u.\, t\ R\ u$

But what is $R$ when we talk about infinite rewriting?

- the single-step rewrite relation, or...
- one of the many transfinite relations (but which?), and ...
- there is also the thorny issue of reflexivity

# The thorny issue of reflexivity

- for finite rewriting, we have that
  - normal forms of R and $R^+$ coincide
  - $R^*$ has no normal forms (w.r.t. our previous definition)
  - what about a variant notion of nf that takes $R^*$ as its starting point?
- we may also have 1-step relations that are naturally reflexive, e.g. developments
- what is normal then?

# Quasi-normal forms

quasi-normal-form, variant I:

- t is a quasi normal form if

$$\forall u.\, t\; R\; u \Rightarrow t = u$$

variant II:

- t is a quasi-normal form if

$$\forall u.\, t\; R\; u \Rightarrow u\; R\; t$$

The latter notion is sometimes used in connection with well-founded quasi-orders

# Rewriting with infinite terms

- there is one argument why the single-step rewrite relation $\to_R$ should be reflexive on infinite terms:
- if $a \to_R b$ then for any term t and position p of t: $t[a]_p \to_R t[b]_p$
- if t is infinite the redex/contractum vanish in the limit, for arbitrarily long p
- thus if we want the relation $\to_R$ to be upper-semi-continuous then we should have $t \to_R t$

# Infinitary Rewriting

- that issue aside, pretty much all our transfinite rewrite relations are reflexive anyway
- we can fiddle with them a little bit to derive versions that do not automatically exhibit reflexivity:
  - the reduction-sequence-based notions (weak reduction, strong reduction, adherence) could request non-empty sequences
  - the notions that use reflexive-transitive closure within their construction (topological closure, pointwise closure, coinductive rewriting) use transitive closure instead

# After this modification...

- ...the sequence-based reductions, as well pointwise closure have the same normal forms as the single-step relation
- but this is not true for:
  - topological closure
  - co-inductive rewriting
  - double-pointwise closure, i.e. the relation is the smallest relation such that both $\rightarrow\!\!\gg$ and $\ll\!\!\leftarrow$ are pointwise closed and transitive
- these other notions “extend reductions to the left”, as well as to the right

# Why extend to the left at all?

- truly symmetric treatment of semantic equality
- well-suited to model construction (our original motivation), in particular w.r.t. orthogonal rewriting

# As a side problem...

- two of these three relations are reflexive on infinite terms
  - the topological closure even of the single-step relation is reflexive on infinite terms, as long as the relation is non-empty
  - when we construct the largest fixpoint for co-inductive rewriting, reflexivity on infinite terms is always preserved (even if $\rightarrow_R = \emptyset$)
  - only in the double-pointwise-closure is this not an issue

# Certain things are no longer qNF

- example 1: $C(A) \to A$, now $C^\infty$ is not a quasi-NF for the left-extended relations
- example 2 (Klop):

  $A \to C(A), C(x) \to D\big(x, C(x)\big), D(x,x) \to E$; the term $D\big(E, D(E, \dots)\big)$ rewrites with all left-extended relations to $E$

  this system has now unique quasi normal forms;

Question: have all non-collapsing non-$\omega$-overlapping systems unique qNFs for these left-extended relations?

# Co-inductive reasoning

- ...about infinite quasi-normal forms:
  - if $\sigma$ is a substitution, mapping variables to qNFs, and...
  - $t$ is the right kind of finite term
  - then $t\sigma$ is a qNF
- But what is the right kind of term? We could use constructor terms, or...

# Pseudo-Constructors

- ...are finite and linear term, such that:
  - it does not unify with any lhs
  - its subterms are either variables or pseudo-constructors
- note:
  - all finite ground NF are pseudo-constructors
  - every constructor is a pseudo-constructor

# What can we do with them?

- given an orthogonal iTRS, turn it into a constructor TRS

1. double-up the signature, each function symbol F has a constructor version $F_c$, and a destructor $F_d$,
2. Functions $\lfloor t \rfloor$ and $\lceil t \rceil$ replace all function symbols in t with their constructor/destructor
3. replace each rule $F(t_1, \dots, t_n) \to r$ with $F_d(\lfloor t_1 \rfloor, \dots, \lfloor t_n \rfloor) \to \lceil r \rceil$
4. For each pseudo-constructor $F(t_1, \dots, t_n)$ add a rule $F_d(\lfloor t_1 \rfloor, \dots, \lfloor t_n \rfloor) \to \lfloor F(t_1, \dots, t_n) \rfloor$

# Resulting System

- is almost orthogonal, and ...
- when we restrict “4” to “minimal” pseudo-constructors then it is a finite and orthogonal constructor iTRS
- its many-step relation restricted to destructor terms is the old many-step relation
- which goes to show that orthogonal rewrite systems are constructor rewrite systems in disguise

# On a side note

- if the system is non-$\omega$-overlapping (but not left-linear), then we can drop the linearity part of pseudo-constructors, and have any finite term which inherently does not unify with lhss as a pseudo-constructor
- the resulting system is almost non-overlapping (but infinite), with the same rewrite theory
- but does it have unique NFs???

# Future Work

- these final question marks go back to our original motivation
- if non-$\omega$-overlapping constructor TRS have unique NFs then this is also the case for arbitrary non-$\omega$-overlapping TRS; but "if"
- one can use this to build normal form models:
  - data are (infinitary) constructor terms
  - infinitary, as substitutions on infinitary constructor terms have a CPO structure (more: Scott-Ershov domain)