PROGRAMMING PEARL

Computing Convex Hulls with a Linear Solver

FLORENCE BENOY and ANDY KING
Computing Laboratory, University of Kent, UK.
email: {p.m.benoy, a.m.king}@kent.ac.uk

FRED MESNARD
Iremia, Université de La Réunion, France.
email: fred@univ-reunion.fr

Abstract

A programming tactic involving polyhedra is reported that has been widely applied in the polyhedral analysis of (constraint) logic programs. The method enables the computations of convex hulls that are required for polyhedral analysis to be coded with linear constraint solving machinery that is available in many Prolog systems.
Keywords: convex hull, polyhedra, abstract interpretation, linear constraints.

1 Introduction

Polyhedra have been widely applied in program analysis (Cousot & Halbwachs, 1978) particularly for reasoning about logic and constraint logic programs. In this context polyhedra have been used in binding-time analysis (Vanhoof & Bruynooghe, 2001), cdr-coded list analysis (Horspool, 1990), argument-size analysis (Benoy & King, 1996), time-complexity analysis (King *et al.*, 1997), high-precision groundness analysis (Codish *et al.*, 2001), type analysis (Sağlam & Gallagher, 1997), termination checking (Codish & Taboch, 1999) and termination inference (Mesnard & Neumerkel, 2001; Genaim & Codish, 2001).

All these techniques use polyhedra to describe relevant properties of the program and manipulate polyhedra using operations that include projection, emptiness checking, inclusion testing for polyhedra, intersection of polyhedra (meet) and the convex hull (join). The classic approach to polyhedral analysis (Cousot & Halbwachs, 1978) uses two representations: (i) frames and rays and (ii) systems of (non-strict) linear inequalities and employs the Chernikova algorithm to convert between them (Le Verge, 1992). The rationale for this dual representation is that the convex hull can be computed straightforwardly with frames and rays whereas intersection is more simply computed over systems of linear inequalities. A simpler tactic that has been widely adopted in the analysis of logic programs is to use only the linear inequality representation and compute the convex hull by adapting (Benoy &

King, 1996) a relaxation technique proposed in (De Backer & Beringer, 1993). The elegance of this approach is that it enables the convex hull to be computed without recourse to a dual representation: the problem is recast as a projection problem that can be subcontracted to standard linear constraint solving machinery with minimal coding effort. Moreover, the performance is acceptable for many applications. In fact this technique has been widely applied in the analysis of logic programs (Codish & Taboch, 1999; Genaim & Codish, 2001; King *et al.*, 1997; Mesnard & Neumerkel, 2001; Sağlam & Gallagher, 1997). The next section outlines the method and the following section, an example implementation. The final section presents the concluding discussion.

2 Method

Consider two arbitrary polyhedra, P_1 and P_2, represented in standard form:

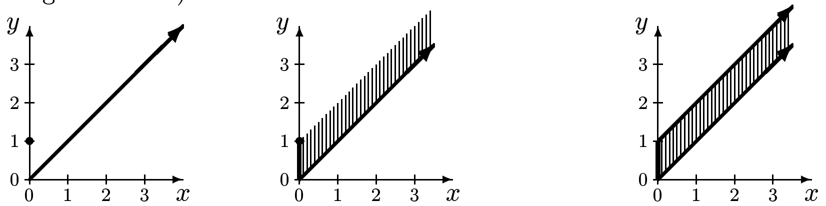
$$P_1 = \{\vec{x} \in \mathbb{Q}^n \mid A_1\vec{x} \leq \vec{B}_1\} \qquad P_2 = \{\vec{x} \in \mathbb{Q}^n \mid A_2\vec{x} \leq \vec{B}_2\}$$

such that $P_1 \neq \emptyset$ and $P_2 \neq \emptyset$ so that the problem is non-trivial. Note that $A_i\vec{x} \leq \vec{B}_i$ are non-strict and therefore P_1 and P_2 are both closed. The problem in essence is to compute the smallest polyhedron that includes P_1 and P_2. Interestingly, the convex hull of $P_1 \cup P_2$ is not necessarily closed as is illustrated in the following example.

Example 2.1
Consider the 2-dimensional polyhedra P_1 and P_2 defined by:

$$P_1 = \left\{ \vec{x} \in \mathbb{Q}^2 \;\middle|\; \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \vec{x} \leq \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\} \quad P_2 = \left\{ \vec{x} \in \mathbb{Q}^2 \;\middle|\; \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -1 & 0 \end{bmatrix} \vec{x} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Observe that $P_1 = \{\langle 0, 1\rangle\}$ is a point whereas $P_2 = \left\{\langle x, y\rangle \in \mathbb{Q}^2 \mid x = y \wedge 0 \leq x\right\}$ is a half-line. Note too that P_1 and P_2 are closed whereas the convex hull of $P_1 \cup P_2$ excludes the points $\left\{\langle x, y\rangle \in \mathbb{Q}^2 \mid x > 0 \wedge y = x + 1\right\}$ and hence is not closed (see the diagram below).



P_1 and P_2 convex hull of $P_1 \cup P_2$ closure of convex hull of $P_1 \cup P_2$

Since the convex hull of $P_1 \cup P_2$ is not necessarily closed, the convex hull cannot always be represented by a system of non-strict linear inequalities; in order to overcome this problem, the closure of the convex hull of $P_1 \cup P_2$ is computed. The starting point for our construction is the convex hull of $P_1 \cup P_2$ that is given by:

$$P_H = \left\{ \vec{x} \in \mathbb{Q}^n \;\middle|\; \begin{array}{lllll} \vec{x} = \sigma_1\vec{x}_1 + \sigma_2\vec{x}_2 & \wedge & \sigma_1 + \sigma_2 = 1 & \wedge & 0 \leq \sigma_1 \quad \wedge \\ A_1\vec{x}_1 \leq \vec{B}_1 & \wedge & A_2\vec{x}_2 \leq \vec{B}_2 & \wedge & 0 \leq \sigma_2 \end{array} \right\}$$

To avoid the non-linearity $\vec{x} = \sigma_1 \vec{x}_1 + \sigma_2 \vec{x}_2$ the system can be reformulated (relaxed) by putting $\vec{y}_1 = \sigma_1 \vec{x}_1$ and $\vec{y}_2 = \sigma_2 \vec{x}_2$ so that $\vec{x} = \vec{y}_1 + \vec{y}_2$ and $A_i \vec{y}_i \leq \sigma_i \vec{B}_i$ to define:

$$P_{CH} = \left\{ \vec{x} \in \mathbb{Q}^n \;\middle|\; \begin{array}{lllll} \vec{x} = \vec{y}_1 + \vec{y}_2 & \wedge & \sigma_1 + \sigma_2 = 1 & \wedge & 0 \leq \sigma_1 \quad \wedge \\ A_1 \vec{y}_1 \leq \sigma_1 \vec{B}_1 & \wedge & A_2 \vec{y}_2 \leq \sigma_2 \vec{B}_2 & \wedge & 0 \leq \sigma_2 \end{array} \right\}$$

Observe that $P_H \subseteq P_{CH}$. Moreover, unlike P_H, P_{CH} is expressed in terms of a system of linear inequalities. Note too that P_{CH} is closed since the projection of a system of non-strict linear inequalities is closed. In fact the following proposition asserts that P_{CH} coincides with the closure of the convex hull of $P_1 \cup P_2$.

Proposition 2.1
P_{CH} is the closure of the convex hull of P_1 and P_2.

The proof uses the concept of a recession cone. The recession cone of a polyhedron P, denoted 0^+P, is defined by: $0^+P = \{\vec{y} \in \mathbb{Q}^n \mid \forall \lambda \geq 0 \, . \, \forall \vec{x} \in P \, . \, \vec{x} + \lambda \vec{y} \in P\}$. The intuition is that 0^+P includes a vector $\vec{y}$ whenever P includes all the half-lines in the direction of $\vec{y}$ that start in P.

Proof
Suppose $P_i = \{\vec{x} \in \mathbb{Q}^n \mid A_i \vec{x} \leq \vec{B}_i\}$. Theorem 19.6 of (Rockafellar, 1970) states that the closure of the convex hull of $P_1 \cup P_2$ is the set $(0^+P_1 + P_2) \cup (P_1 + 0^+P_2) \cup (\cup\{\sigma_1 P_1 + \sigma_2 P_2 \mid \sigma_1 + \sigma_2 = 1 \wedge 0 < \sigma_1, \sigma_2\})$. Intuitively, $0^+P_1 + P_2$ is P_2 extended in the directions of half-lines contained within P_1. Let $\vec{x} \in P_i$, then $\vec{y} \in 0^+P_i$ if and only if $A_i(\vec{x} + \lambda \vec{y}) \leq \vec{B}_i$ for all $\lambda \geq 0$ which holds if and only if $A_i \vec{y} \leq \vec{0}$ (Rockafellar, 1970)[pp 62]. Therefore $0^+P_1 + P_2 = \{\vec{x} \in \mathbb{Q}^n \mid \vec{x} = \vec{y}_1 + \vec{y}_2 \wedge A_1 \vec{y}_1 \leq \vec{0} \wedge A_2 \vec{y}_2 \leq \vec{B}_2\}$ and similarly $P_1 + 0^+P_2 = \{\vec{x} \in \mathbb{Q}^n \mid \vec{x} = \vec{y}_1 + \vec{y}_2 \wedge A_1 \vec{y}_1 \leq \vec{B}_1 \wedge A_2 \vec{y}_2 \leq \vec{0}\}$. Furthermore, $\cup\{\sigma_1 P_1 + \sigma_2 P_2 \mid \sigma_1 + \sigma_2 = 1 \wedge 0 < \sigma_1, \sigma_2\} = \{\vec{x} \in \mathbb{Q}^n \mid \sigma_1 + \sigma_2 = 1 \wedge 0 < \sigma_1, \sigma_2 \wedge \vec{x} = \vec{y}_1 + \vec{y}_2 \wedge A_1 \vec{y}_1 \leq \sigma_1 \vec{B}_1 \wedge A_2 \vec{y}_2 \leq \sigma_2 \vec{B}_2\}$. Observe that $\{\vec{x} \in \mathbb{Q}^n \mid \vec{x} = \vec{y}_1 + \vec{y}_2 \wedge A_1 \vec{y}_1 \leq \sigma_1 \vec{B}_1 \wedge A_2 \vec{y}_2 \leq \sigma_2 \vec{B}_2\}$ coincides with the sets (i) $0^+P_1 + P_2$, (ii) $P_1 + 0^+P_2$ and (iii) $\cup\{\sigma_1 P_1 + \sigma_2 P_2 \mid \sigma_1 + \sigma_2 = 1 \wedge 0 < \sigma_1, \sigma_2\}$ when (i) $\sigma_1 = 0$ and $\sigma_2 = 1$, (ii) $\sigma_1 = 1$ and $\sigma_2 = 0$ and (iii) $\sigma_1 + \sigma_2 = 1$ and $0 < \sigma_1, \sigma_2$ respectively. Therefore P_{CH} is the closure of the convex hull. □

This result leads to an algorithm for computing the closure of the convex hull: construct the systems $A_i \vec{y}_i \leq \sigma_i \vec{B}_i$ by scaling the constant vectors $\vec{B}_i$ by σ_i, add the constraints $\vec{x} = \vec{y}_1 + \vec{y}_2$, $\sigma_1 + \sigma_2 = 1$ and $0 \leq \sigma_i$, then eliminate variables other than $\vec{x}$ using projection to obtain P_{CH} in terms of $\vec{x}$. Hence the closure of the convex hull can be computed without recourse to another representation. This is illustrated below.

Example 2.2
Returning to example 2.1, consider the systems $A_i \vec{x} \leq \vec{B}_i$:

$$P_1 = \left\{ \langle x, y \rangle \in \mathbb{Q}^2 \;\middle|\; \begin{array}{l} x \leq 0 \wedge -x \leq 0 \quad \wedge \\ y \leq 1 \wedge -y \leq -1 \end{array} \right\} \quad P_2 = \left\{ \langle x, y \rangle \in \mathbb{Q}^2 \;\middle|\; \begin{array}{r} x - y \leq 0 \wedge \\ -x + y \leq 0 \wedge \\ -x \leq 0 \end{array} \right\}$$

Adding $\vec{x} = \vec{y}_1 + \vec{y}_2$, $\sigma_1 + \sigma_2 = 1$ and $0 \leq \sigma_i$ leads to the following system:

$$P_{CH} = \left\{ \langle x, y \rangle \in \mathbb{Q}^2 \;\middle|\; \begin{array}{lclcl} x = x_1 + x_2 & \wedge & y = y_1 + y_2 & \wedge & \sigma_1 + \sigma_2 = 1 \;\; \wedge \\ 0 \leq \sigma_1 & \wedge & 0 \leq \sigma_2 & \wedge & \\ x_1 \leq 0 & \wedge & -x_1 \leq 0 & \wedge & \\ y_1 \leq \sigma_1 & \wedge & -y_1 \leq -\sigma_1 & \wedge & \\ x_2 - y_2 \leq 0 & \wedge & -x_2 + y_2 \leq 0 & \wedge & -x_2 \leq 0 \end{array} \right\}$$

Eliminating the variables x_i, y_i and σ_i leads to the solution:

$$P_{CH} = \{\langle x, y \rangle \in \mathbb{Q}^2 \mid 0 \leq x \wedge x \leq y \wedge y \leq x + 1\}$$

Theorem 19.6 of (Rockafellar, 1970), which is used in the proof, asserts that P_{CH} includes $P_1 + 0^+ P_2 = P_1 + P_2 = \{\langle x, y \rangle \in \mathbb{Q}^2 \mid x \geq 0 \wedge y = x + 1\}$ and therefore includes the points $\{\langle x, y \rangle \in \mathbb{Q}^2 \mid x > 0 \wedge y = x + 1\}$, and hence ensures closure. Note that calculating P_{CH} without the inequalities $0 \leq \sigma_1$ and $0 \leq \sigma_2$ – the relaxation advocated in (De Backer & Beringer, 1993) for computing convex hull – gives $\{\langle x, y \rangle \in \mathbb{Q}^2 \mid 0 \leq x\}$ which is incorrect.

3 Implementation

This section shows how closure of the convex hull can be implemented elegantly using a linear solver in particular the CLP($\mathbb{Q}$) library (Holzbaur, 1995). The behaviour of a predicate is described with the aid of modes, that is, + indicates an argument that should be instantiated to a non-variable term when the predicate is called; - indicates an argument that should be uninstantiated; and ? indicates an argument that may or may not be instantiated (Deransart *et al.*, 1996).

3.1 Closed Polyhedra

Closed polyhedra will be represented by lists (conjunctions) of linear constraints of the form $c ::= e \leq e \mid e = e \mid e \geq e$ where expressions take the form $e ::= x \mid n \mid n * x \mid -e \mid e + e \mid e - e$ and n is a rational number and x is a variable. A convenient representation for a closed polyhedron is a (non-ground) list of constraints. This representation is interpreted with respect to a totally ordered (finite) set of variables. The ordering governs the mapping of each variable to its specific dimension. In practise, the ordering on variables is itself represented by the position of each variable within a list. Specifically, if C is a list of linear constraints $[c_1, \ldots, c_m]$ and X is a list of variables $[x_1, \ldots, x_n]$, then the represented polyhedron is $P_{C,X} = \{\langle y_1, \ldots, y_n \rangle \in \mathbb{Q}^n \mid (\wedge_{i=1}^n x_i = y_i) \models_{\mathbb{Q}} (\wedge_{j=1}^m c_j)\}$. Note that although the order of variables in X is significant, the order of the constraints in C is not. Finally, let $vars(o)$ denote the set of variables occurring in the syntactic object o.

Example 3.1
The polyhedron P_1 from example 2.2 can be represented by the lists $C_1 = [x = 0, y = 1]$ and $X = [x, y]$, that is, $P_1 = P_{C_1,X}$. Moreover, $P_2 = P_{C_2,X}$ where $C_2 = [x = y,\ x \geq 0]$ or alternatively $C_2 = [y + z \geq x, x \geq y + 2 * z, y \geq 0, z \geq 0]$.

Hence the dimension of $P_{C,X}$ is defined by the length of the list X rather than the number of variables in C.

3.2 Projection

Projection is central to computing the convex hull. The desire, therefore, is to construct a predicate `project(+Xs,+Cxs,-ProjectCxs)` that is true when for a given list of dimensions `Xs` and a given list of constraints `Cxs`, `ProjectCxs` is the projection of `Cxs` onto `Xs`. The specification of such a predicate is given below.

preconditions:

- `Xs` is a closed list with distinct variables as elements,
- `Cxs` is a closed list of linear constraints,
- `Cxs` is satisfiable.

postconditions:

- `Xs` is a closed list with distinct variables as elements,
- `ProjectCxs` is a closed list of linear constraints,
- $vars(\texttt{ProjectCxs}) \subseteq vars(\texttt{Xs})$,
- $P_{\texttt{Cxs},\texttt{Xs}} = P_{\texttt{ProjectCxs},\texttt{Xs}}$.

Such a predicate can be constructed by adding the given constraints to the store and then invoking the projection facility provided in the CLP($\mathbb{Q}$) library, that is, the predicate `dump(+Target, -NewVars, -CodedAnswer)` (Holzbaur, 1995). Quoting from the manual: "[dump] reflects the constraints on the target variables into a term, where `Target` and `NewVars` are lists of variables of equal length and `CodedAnswer` is the term representation of the projection of constraints onto the target variables where the target variables are replaced by the corresponding variables from `NewVars`". This leads to the following implementation of `project`:

```
:- use_module(library(clpq)).

project(Xs, Cxs, ProjectCxs) :-
    tell_cs(Cxs),
    dump(Xs, Vs, ProjectCxs), Xs = Vs.

tell_cs([]).
tell_cs([C|Cs]) :- {C}, tell_cs(Cs).
```

Example 3.2
For example, the query `project([X, Z], [X < Y, Y < Z], ProjectCs)` will correctly bind `Cs` to `[X-Z<0]`. However, correctness of this predicate is compromised by existing constraints in the store. For instance, the compound query `{X = Z + 1}`, `project([X, Z], [X < Y, Y < Z], ProjectCs)` will fail because constraints posted within `tell_cs` interact with those already in the store.

To insulate the constraints posted in `tell_cs`, both the variables `Xs` and the constraints `Cxs` need to be renamed. Renaming is trivial with the builtin `copy_term` but care must be taken to ensure that `Xs` and `Cxs` are renamed consistently, that is that variable sharing in `Xs` and `Cxs` is preserved in the copies. However, in SICStus Prolog `copy_term(Term, Cpy)` copies any constraints in the store that involve variables in `Term`. For example, the query {`X=Y`}, `copy_term(X=Y+1, Cpy)` will bind `Cpy` to `_A=_B+1` where `_A` and `_B` are fresh variables. It will also copy the constraint `X = Y` by posting the new constraint `_A = _B` to the store. To nullify this effect, `copy_term` is called within the scope of `call_residue`. The call `call_residue(copy_term(X=Y+1, Cpy), Residue)` residuates any new constraint into `Residue` instead of posting it to the store, thereby copying the term without copying any constraint. Whether residuation is required depends on the particular Prolog system. This leads to the following (SICStus Prolog specific) revision:

```
project(Xs, Cxs, ProjectCxs) :-
    call_residue(copy_term(Xs-Cxs, CpyXs-CpyCxs), _),
    tell_cs(CpyCxs),
    dump(CpyXs, Vs, ProjectCxs), Xs = Vs.
```

Example 3.3
Using this revision, the query {`X = Z + 1`}, `project([X, Z], [X < Y, Y < Z], ProjectCs)` will succeed binding `ProjectCs` to `[X-Z<0]`. However, adding `Z = 5` to the list of constraints induces an error. The problem is that posting the constraints binds `Z` to 5 so that `dump` is called with its first argument instantiated to a list that contains a non-variable term.

A pre-processing predicate `prepare_dump` is therefore introduced to ensure that `dump` is called correctly. The following revision to `project`, in effect, extends the facility provided by `dump` to capture constraints over both uninstantiated and instantiated variables:

```
project(Xs, Cxs, ProjectCxs) :-
    call_residue(copy_term(Xs-Cxs, CpyXs-CpyCxs), _),
    tell_cs(CpyCxs),
    prepare_dump(CpyXs, Xs, Zs, DumpCxs, ProjectCxs),
    dump(Zs, Vs, DumpCxs), Xs = Vs.

prepare_dump([], [], [], Cs, Cs).
prepare_dump([X|Xs], YsIn, ZsOut, CsIn, CsOut) :-
    (ground(X) ->
        YsIn  = [Y|Ys],
        ZsOut = [_|Zs],
        CsOut = [Y=X|Cs]
    ;
        YsIn  = [_|Ys],
        ZsOut = [X|Zs],
        CsOut = Cs
    ),
    prepare_dump(Xs, Ys, Zs, CsIn, Cs).
```

The literal `prepare_dump(+Xs, +Ys, -Zs, ?CsIn, -CsOut)` is true for a given list `Xs` which contains either variables or numbers (or a mixture of the two) and a given list `Ys` which contains only variables, if

- `Zs` is the list obtained by substituting the non-variable terms of `Xs` with fresh variables and
- `CsOut` is an open ended list of equality constraints with `CsIn` at its end that contains one equality constraint for each number in `Xs`. Each constraint equates a numeric element of `Xs` with the element of `Ys` that is in the same list position.

The call `prepare_dump([X1, 1, X3, 2], [A, B, C, D], Zs, CsIn, CsOut)`, for instance, will bind `Zs` to `[X1,_A,X3,_B]` and `CsOut` to `[B=1,D=2|CsIn]`. The predicate ensures that `dump` is called with its first argument bound to a list of free variables even when the list `Xs` includes numbers. In the CLP($\mathbb{Q}$) library, numbers coincide with rationals which are represented as compound (ground) terms of the form `rat(`n`, `d`)` where n and d are integers. The `ground(X)` test effectively checks whether `X` is instantiated to a number; the test `number(X)` is inappropriate since it would always fail.

Example 3.4
Consider again example 3.1. The second representation of P_2 can be simplified by using projection as follows:

```
| ?- Cs = [Y+Z>=X,X>=Y+2*Z,Y>=0,Z>=0], project([X,Y], Cs, ProjectCs).
ProjectCs = [Y>=0,X=Y] ? ;
no
```

The system `Cs` is expressed over 3 variables and therefore defines a 3 dimensional space. Intuitively, the projection onto `[X, Y]` is the shadow cast by $P_{\texttt{Cs},[\texttt{X},\texttt{Y},\texttt{Z}]}$ onto the 2 dimensional space over `X` and `Y`. The projection `ProjectCs` in fact defines a half-line confined to the first quadrant since, by rearranging `Cs`, it follows that $P_{\texttt{Cs},[\texttt{X},\texttt{Y},\texttt{Z}]} = \{\langle x, y, z\rangle \in \mathbb{Q}^3 \mid x = y \wedge 0 \leq y \wedge z = 0\}$.

3.3 Convex Hull

The specification for the main predicate `convex_hull(+Xs, +Cxs, +Ys, +Cys, -Zs, -Czs)`, and then its code, is given below.

preconditions:

- `Xs` is a closed list with distinct variables as elements and likewise for `Ys`,
- `Xs` and `Ys` have the same length,
- $vars(\texttt{Xs}) \cap vars(\texttt{Ys}) = \emptyset$,
- `Cxs` and `Cys` are closed lists of linear constraints,
- `Cxs` and `Cys` are both satisfiable,
- $vars(\texttt{Cxs}) \subseteq vars(\texttt{Xs})$ and $vars(\texttt{Cys}) \subseteq vars(\texttt{Ys})$.

postconditions:

- Xs, Ys and Zs are closed lists with distinct variables as elements,
- Zs is the same length as both Xs and Ys,
- Czs is a closed list of linear constraints,
- $vars(\texttt{Czs}) \subseteq vars(\texttt{Zs})$ and $(vars(\texttt{Xs}) \cup vars(\texttt{Ys})) \cap vars(\texttt{Zs}) = \emptyset$,
- $P_{\texttt{Czs},\texttt{Zs}}$ is the closure of the convex hull of $P_{\texttt{Cxs},\texttt{Xs}} \cup P_{\texttt{Cys},\texttt{Ys}}$.

```
convex_hull(Xs, Cxs, Ys, Cys, Zs, Czs) :-
    scale(Cxs, Sig1, [], C1s),
    scale(Cys, Sig2, C1s, C2s),
    add_vect(Xs, Ys, Zs, C2s, C3s),
    project(Zs, [Sig1 >= 0, Sig2 >= 0, Sig1+Sig2 = 1|C3s], Czs).

scale([], _, Cs, Cs).
scale([C1|C1s], Sig, C2s, C3s) :-
    C1 =.. [RelOp, A1, B1],
    C2 =.. [RelOp, A2, B2],
    mul_exp(A1, Sig, A2),
    mul_exp(B1, Sig, B2),
    scale(C1s, Sig, [C2|C2s], C3s).

mul_exp(E1, Sigma, E2) :- once(mulexp(E1, Sigma, E2)).

mulexp(  X,    _,      X) :- var(X).
mulexp(N*X,    _,    N*X) :- ground(N), var(X).
mulexp( -X, Sig,      -Y) :- mulexp(X, Sig, Y).
mulexp(A+B, Sig,     C+D) :- mulexp(A, Sig, C), mulexp(B, Sig, D).
mulexp(A-B, Sig,     C-D) :- mulexp(A, Sig, C), mulexp(B, Sig, D).
mulexp(  N, Sig, N*Sig) :- ground(N).

add_vect([], [], [], Cs, Cs).
add_vect([U|Us], [V|Vs], [W|Ws], C1s, C2s) :-
    add_vect(Us, Vs, Ws, [W = U+V|C1s], C2s).
```

The predicate `mulexp(?E1, ?Sigma, -E2)` scales the numeric constants that occur within `E1` by the variable `Sigma`, providing they are not coefficients of variables, to obtain the expression `E2`. Note that `Sigma` is a variable and the expression `E1` may be a variable, hence both `E1` and `Sigma` have mode ? rather than +. Since a non-ground representation is employed for expressions, the test `var(X)` is used to determine whether the expression is a variable. As before, the test `ground(N)` detects numeric constants – rational numbers – which are the only type of subexpressions that are ground. Observe that `mulexp` can return more than one solution, for example, `mulexp(X, Sig, E2)` generates `E2 = X`; `X = -(_A), E2 = -(_A)`; `X = -(-(_A)), E2 = -(-(_A))` etc as solutions. Thus the pruning operator `once` is applied within `mul_exp(?E1, ?Sigma, -E2)` to prevent erroneous solutions.

The predicate `scale(+C1s, ?Sigma, ?C2s, -C3s)` scales each constraint within the list `C1s` by the variable `Sigma`. Each constraint consists of a binary operator and two expressions, and scaling is applied to the numeric constants in each expression as specified by `mul_exp`. For example, `scale([X+2 >= 1+Y, Y = Z], Sigma, Tail, ScaledCs)` binds `ScaledCs` to `[Y = Z, X+2*Sigma >= 1*Sigma+Y | Tail]`. Note that `scale` finesses the problem of putting `Cxs` and `Cys` into the standard form $A_i\vec{y}_i \leq \vec{B}_i$ before applying scaling. In standard form, `X+2 >= 1+Y` is `Y-X =< 1` but scaling constants on both sides of the relational operator preserves equivalence in that `X+2*Sig >= 1*Sig+Y` is equivalent to `Y-X =< 1*Sig`. The use of a difference list avoids an unnecessary call to append in the body of `convex_hull`.

The predicate `add_vect(+Us, +Vs, -Ws, ?C1s, -C2s)` operates on the lists `Us` `=` `[`U_1`, ..., `U_n`]` and `Vs = [`V_1`, ..., `V_n`]` which correspond to the vectors $\vec{y}_1$ and $\vec{y}_2$ (as introduced in section 2). The argument `Ws` is instantiated to another list of variables `[`W_1`, ..., `W_n`]`, which corresponds with $\vec{x}$. The predicate creates the system of equalities `[`W_1 `=` U_1`+`V_1`, ..., `W_n `=` U_n`+`V_n`]` corresponding to the system $\vec{x} = \vec{y}_1 + \vec{y}_2$. The scaled constraints output by the two calls to `scale` are passed to `add_vect` via its accumulator and thereby combined with the system of equalities. For example, the call `add_vect([X1,Y1], [X2, Y2], Ws, Tail, Cs)` returns the bindings `Cs = [_A=Y1+Y2,_B=X1+X2|Tail]` and `Ws = [_B,_A]`.

The predicate `convex_hull(Xs, Cxs, Ys, Cys, Zs, Czs)` takes, as input, two lists of constraints (`Cxs` and `Cys`) and their corresponding lists of variables (`Xs` and `Ys`) and produces as output a single list of constraints `Czs` over the variables `Zs` that represents the closure of the convex hull of the two input polyhedra. If `Xs` and `Ys` are not variable disjoint, then the pre-requisite can be satisfied by appropriately renaming variables. Specifically, the variables `Xs` and constraints `Cxs` can be renamed with `copy_term(Xs-Cxs, CpyXs-CpyCxs)` and the call `convex_hull(Xs, Cxs, Ys, Cys, Zs, Czs)` replaced with `convex_hull(CpyXs, CpyCxs, Ys, Cys, Zs, Czs)`. Since the integrity of the constraint store is preserved by `project` and since `project` is the only source of interaction with the store, then it follows that `convex_hull` also does not side-effect any existing constraints. The following is an illustrative example.

Example 3.5
Running this code on the data of Example 2.2 gives:

```
| ?- convex_hull([X1,Y1],[X1=0,Y1=1],[X2,Y2],[X2>=0,Y2=X2],V,S).
S = [_A>=0,_A-_B>=-1,_A-_B=<0],
V = [_A,_B] ? ;
no
```

4 Discussion

This section discusses the method proposed in the paper, comparing it with related techniques. The Chernikova method is exponential in the worst-case (Le Verge, 1992) and the Fourier-Motzkin method, like all projection techniques over linear

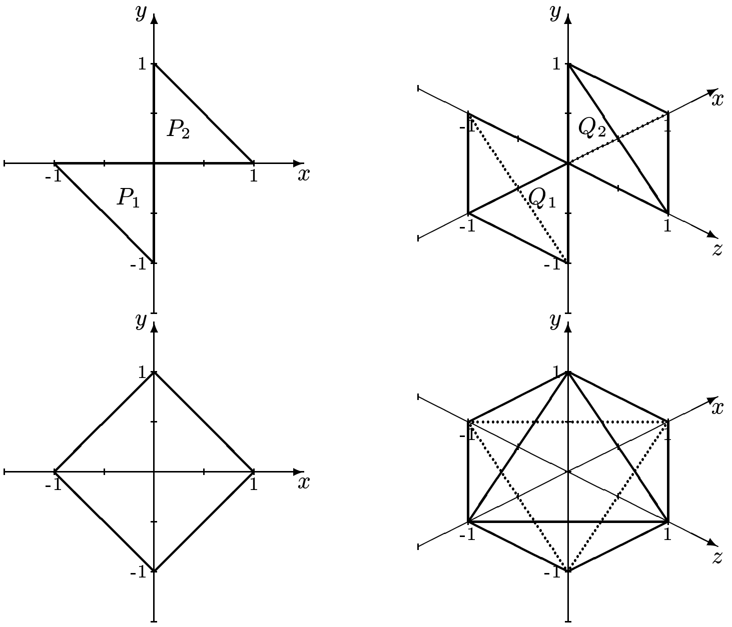


Fig. 1. (i) P_1 and P_2, (ii) Q_1 and Q_2, (iii) $conv(P_1 \cup P_2)$, (iv) $conv(Q_1 \cup Q_2)$

inequalities (Chandru *et al.*, 2000), is also exponential. The exponential behaviour of both methods stems from the same source: the possibly exponential relationship between the number of vertices and the number of half-spaces that define a polyhedron. In fact the problem of calculating the closure of the convex hull of two polyhedra is also exponential even for bounded polyhedra (polytopes). This can be demonstrated by considering the so-called cross polytope in n-dimensions which is the polyhedron with the vertex set $\{\langle \pm 1, 0, \ldots, 0\rangle, \langle 0, \pm 1, \ldots, 0\rangle, \ldots, \langle 0, 0, \ldots, \pm 1\rangle\}$. The cross polytope can be defined by no less than 2^n inequalities yet can arise as the convex hull of two polyhedra both of which can be defined with $O(n)$ inequalities. Specifically consider the n-dimensional polyhedra

$$P_1 = \{\langle x_1, \ldots, x_n\rangle \in \mathbb{Q}^n \mid (\textstyle\sum_{i=1}^{n} -x_i \leq 1) \wedge (\wedge_{j=1}^{n} x_j \leq 0)\}$$
$$P_2 = \{\langle x_1, \ldots, x_n\rangle \in \mathbb{Q}^n \mid (\textstyle\sum_{i=1}^{n} x_i \leq 1) \wedge (\wedge_{j=1}^{n} - x_j \leq 0)\}$$

Because P_1 and P_2 are polytopes, they can be expressed in terms of their vertices:

$$P_1 = conv(\{\langle 0, 0, \ldots, 0\rangle, \langle -1, 0, \ldots, 0\rangle, \langle 0, -1, \ldots, 0\rangle, \ldots, \langle 0, 0, \ldots, -1\rangle\})$$
$$P_2 = conv(\{\langle 0, 0, \ldots, 0\rangle, \quad \langle 1, 0, \ldots, 0\rangle, \quad \langle 0, 1, \ldots, 0\rangle, \ldots, \langle 0, 0, \ldots, 1\rangle\})$$

Since $\langle 0, 0, \ldots, 0\rangle$ is convexly spanned by $\langle 1, 0, \ldots, 0\rangle$ and $\langle -1, 0, \ldots, 0\rangle$, it follows that $cl(conv(P_1 \cup P_2)) = conv(P_1 \cup P_2) = conv(\{\langle \pm 1, 0, \ldots, 0\rangle, \langle 0, \pm 1, \ldots, 0\rangle, \ldots,$ $\langle 0, 0, \ldots, \pm 1\rangle\})$ which is the n-dimensional cross polytope. The 2 and 3 dimensional cases are denoted in Figure 1 by (i) P_1 and P_2 and (ii) Q_1 and Q_2 respectively for which the cross polytopes are a solid square and an octahedron. Hence the problem of calculating the closure of the convex hull is intrinsically exponential irrespective of the algorithm employed.

Example 4.1

The following query illustrates how the hull algorithm yields an exponential number of inequalities for the 4 dimensional case.

```
| ?- Xs = [X1, X2, X3, X4], Ys = [Y1, Y2, Y3, Y4],
     Cxs = [-1 =< X1+X2+X3+X4, X1 =< 0, X2 =< 0, X3 =< 0, X4 =< 0],
     Cys = [ Y1+Y2+Y3+Y4 =< 1, 0 =< Y1, 0 =< Y2, 0 =< Y3, 0 =< Y4],
     convex_hull(Xs, Cxs, Ys, Cys, Zs, Czs),
     Zs = [A, B, C, D].

Czs = [A-B+C+D>=-1, A+B-C-D=<1, A+B+C+D>=-1, A-B-C-D=<1,
       A-B-C+D>=-1, A+B+C-D=<1, A+B-C+D>=-1, A-B+C-D=<1,
       A-B+C-D>=-1, A+B-C+D=<1, A+B+C-D>=-1, A-B-C+D=<1,
       A-B-C-D>=-1, A+B+C+D=<1, A+B-C-D>=-1, A-B+C+D=<1] ? ;

no
```

However, it would be wrong to conclude from these examples that the frame and ray representation is preferable – inequalities are unavoidable since they are required for other polyhedral operations.

Despite the scaling problems that are inherent to any convex hull algorithm, in practise the technique proposed in this paper has been widely applied in logic programming (Codish & Taboch, 1999; Genaim & Codish, 2001; King *et al.*, 1997; Mesnard & Neumerkel, 2001; Sağlam & Gallagher, 1997), mostly to satisfaction. For example, in the context of inferring termination conditions for logic programs this method is feasible since it accounts for 42% of this first pass of the analysis and the first pass itself constitutes only 23% of the total analysis time (Mesnard & Neumerkel, 2001). Whether the approach presented in this paper is applicable depends on the application context. When only standard domain operations are required and performance is not an issue, this method has much to commend it. However, when the application has to additionally reason, say, about integral points (Ancourt, 1991; Quinton *et al.*, 1997) or parameterised polyhedra (Loechner & Wilde, 1997) then specialised polyhedral libraries are required. Further, if performance is important, then recourse should be made to a polyhedral library, since a state-of-the-art implementation employing the Chernikova algorithm (Bagnara *et al.*, 2002), will outperform the approach presented here.

We have presented a Prolog program for computing convex hulls using linear solver machinery. As Holzbaur's library is also available for CIAO Prolog, ECLiPSe, XSB and Yap Prolog, the technique can be easily adapted to these systems. The method is a reasonable compromise between conciseness, clarity and efficiency and variants of this program have now been widely deployed.

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